# Lecture 2: Representations and group actions

For each non-empty set $S$ we have the permutation group

$$\mathrm{Perm}(S) := \{ S \xrightarrow{f} S \mid f \text{ is bijective} \},$$

where the group structure is given by

$$\begin{aligned} \mathrm{Perm}(S) \times \mathrm{Perm}(S) &\longrightarrow \mathrm{Perm}(S) \\ (f, g) &\longmapsto f \circ g \end{aligned}$$

Here by definition $\forall\, x \in S : (f \circ g)(x) = f(g(x))$.

The class of permutation groups Perm(S) is large enough as to "yield", up to isomorphism, all possible groups. More precisely, we have the following.

<u>Thm</u> (Cayley) Every group $G$ is isomorphic to a subgroup of some permutation group.

In the next section we shall develop the tools to prove this theorem and then prove it.

A representation of a group $G$ on a set $S$ is a group homomorphism

$$\begin{aligned} G &\xrightarrow{\rho} \mathrm{Perm}(S) \\ g &\longmapsto \rho_g \end{aligned}$$

<u>Definition</u> The regular representation is the one where $S = G$ and $\forall\ g \in G$

$$\begin{aligned} G &\xrightarrow{\rho_g} G \\ x &\longmapsto \rho_g(x) := gx \end{aligned}$$

## Proof of Cayley's theorem

Consider the regular representation (so $G = S$). Suppose that $g \in \ker(\rho)$, i.e. $\rho_g = 1_{\mathrm{Perm}(G)}$ so $\forall\, x \in G$:

$$\rho_g(x) = x,$$

$$gx = x,$$

$$g = 1_G.$$

Thus $\ker(\rho) = \{1_G\}$. The first isomorphism thm applied to the homomorphism $\rho$ yields $G \hookrightarrow \mathrm{Perm}(G)$ $\square$

Another useful representation is *conjugation*

$$
\begin{aligned}
G &\xrightarrow{\rho} \mathrm{Aut}(G) \\
g &\longmapsto \left( \begin{array}{c} G \xrightarrow{\rho_g} G \\ x \longmapsto g x g^{-1} \end{array} \right)
\end{aligned}
$$

Indeed, $\forall \; g, x, y \in G$ :

$$\rho_g(xy) = g x y g^{-1} = g x g^{-1} g y g^{-1} = \rho_g(x)\, \rho_g(y),$$

so $\rho_g \in \mathrm{Aut}(G)$, and $\forall \; g_1, g_2, x \in G$

$$\rho_{g_1 g_2}(x) = g_1 g_2 x (g_1 g_2)^{-1} = g_1 g_2 x g_2^{-1} g_1^{-1} = \left( \rho_{g_1} \circ \rho_{g_2} \right)(x),$$

so $\rho$ is a group homomorphism.

A linear representation of $G$ is a group homomorphism

$$G \xrightarrow{\ \rho\ } \mathrm{Aut}(V), \quad V \text{ a } k\text{-vector space}$$

Examples The sign rep's: 
$$\begin{aligned} S_n &\longrightarrow GL_1(k) \\ \pi &\longmapsto \mathrm{sign}(\pi) \end{aligned}$$
and the

permutation rep's

$$S_n \longrightarrow \mathrm{Aut}(V), \quad V = \mathbb{C}^n$$

$$\pi \longmapsto \left( \alpha_1 e_1 + \dots + \alpha_n e_n \mapsto \alpha_1 e_{\pi(1)} + \dots + \alpha_n e_{\pi(n)} \right)$$

Given a group $G$ and a set $S$, an action of $G$ on $S$ is a map

$$\begin{aligned} G \times S &\longrightarrow S \\ (g, s) &\longmapsto g \cdot s \end{aligned}$$

s.t.

(A1) $\forall\, s \in S:\ 1_G \cdot s = s$

(A2) $\forall\, g_1, g_2 \in G,\ s \in S:\ (g_1 g_2) \cdot s = g_1 \cdot (g_2 \cdot s)$.

We may attach to it the representation

$$\begin{aligned} G &\xrightarrow{\ \rho\ } \mathrm{Perm}(S) \\ g &\longmapsto \left( \begin{aligned} S &\xrightarrow{\ \rho_g\ } S \\ x &\longmapsto g \cdot s \end{aligned} \right) \end{aligned}$$

Indeed, $\forall\ g_1, g_2 \in G,\ x \in S$ we have

$$\rho_{g_1 g_2}(x) = (g_1 g_2)\cdot x = g_1 \cdot (g_2 \cdot x) = (\rho_{g_1} \circ \rho_{g_2})(x),$$

so $\rho$ is a group homomorphism.

Conversely, a rep'n $G \xrightarrow{\rho} \mathrm{Perm}(S)$ yields the action

$$\begin{aligned} G \times S &\longrightarrow S \\ (g, x) &\longmapsto g\cdot x := \rho_g(x) \end{aligned}$$

This is indeed an action as $1_G \cdot x = \rho_{1_G}(x) = x$, $\forall\ x \in S$ and $(g_1 g_2)\cdot x = \rho_{g_1 g_2}(x) = (\rho_{g_1} \circ \rho_{g_2})(x)$
$= \rho_{g_1}(\rho_{g_2}(x)) = g_1 \cdot (g_2 \cdot x)$.

In fact, the actions of $G$ on $S$ and the representations of $G$ on $S$ are equivalent theories.

<u>Defn</u> For each $s \in S$ the orbit of $s$ w.r.t. $G$ is the set

$$G \cdot s := \{ g \cdot s : g \in G \} \subseteq S$$

and

$$G_s := \{ g \in G : gs = s \} \leq G$$

is known as the isotropy group of $s$ w.r.t. $G$.

<u>Ex</u> Show that $G_s$ is a subgroup of $G$.

Prop'n For each $s \in S$ and $g \in G$ we have

$$g G_s g^{-1} = G_{gs}$$

Proof

($\subseteq$): Pick $y \in g G_s g^{-1}$. Then $y = g h g^{-1}$, for some $h \in G_s$. But $y \cdot s' = (g h g^{-1})(g \cdot s) = gh \cdot s = gs = s'$, so $y \in G_{s'}$.

($\supseteq$): Now pick $y \in G_{s'}$. We have $y \cdot s' = s'$, $y \cdot (g \cdot s) = g \cdot s$, $g^{-1} \cdot (y \cdot (g \cdot s)) = s$, $(g^{-1} y g) \cdot s = s$. Hence $g^{-1} y g \in G_s$ and then $y \in g G_s g^{-1}$ $\square$

Prop'n Fix $s \in S$. Then we have a bijection

$$G/G_s \xrightarrow{\sim} G \cdot s,$$
$$gG_s \longmapsto g \cdot s$$

Proof

[Ex.]

Note that for each $s, s' \in S$ the corresponding orbits $G \cdot s$ and $G \cdot s'$ are either disjoint or $G \cdot s = G \cdot s'$. Indeed, $G s \cap G s' \neq \emptyset \Rightarrow \exists g_1, g_2 \in G$ s.t.

$$g_1 s = g_2 s'$$

then $g s = s'$, where $g := g_2^{-1} g_1$. Hence

$$G \cdot (g \cdot s) = G \cdot s',$$

$$G s = G s'.$$

This means that there is a disjoint union decomposition

$$S = \coprod_{i \in I} G \cdot s_i \quad ,$$

where the $s_i$'s are elements in distinct orbits. So

$$|S| = \sum_{i \in I} (G : G \cdot s_i)$$

if $|S| < \infty$.

Defn The centre $Z(G)$ of a group $G$ is the set

$$Z(G) = \{ x \in G \text{ s.t. } \forall g \in G : gx = xg \}.$$

Note that $x \in Z(G) \iff |G \cdot x| = 1$, where the action is conjugation. Hence the class formula

$$|G| = |Z(G)| + \sum_{x \in C'} [G : G_x],$$

where $C'$ is the set of conjugacy classes of size $> 1$.