# Lecture 3 Cauchy's theorem

Let $p$ be a prime number.

**Defn** A $p$-group is a group $G$ s.t. $|G| = p^k$, $k \in \{1, 2, 3, \dots\}$.

Suppose $G$ is any finite group and $|G| = p_1^{\alpha_1} \cdot \dots \cdot p_s^{\alpha_s}$, were $p_1, \dots, p_s$ are all distinct primes.

**Defn** If $H \leq G$ s.t. $|H| = p_i^{\alpha_i}$, for some $i \in \{1, \dots, g\}$, then $H$ is a $p_i$-Sylow subgroup of $G$.

Thm Notation as above, $\forall i \in \{1, \dots, s\}$ $\exists$ a $p_i$-Sylow subgroup of $G$.

Before proving the theorem we need the following.

Lemma Let $K$ be a finite abelian group and a prime $p \mid |K|$. Then $\exists$ $H \leq K$ s.t. $|H| = p$.

Proof of Lemma — by induction on the order of $|K|$

If $|K| = p$ the assertion is trivial. So we may assume that $|K| > p$. Let $g \in K$ s.t. $g \neq 1$ and put $H := g^{\mathbb{Z}}$.

Case 1 $p \mid |H|$ : Then

$$x = g^{\frac{|H|}{p}}$$

has order $p$ and we are done.

Case 2 $p \nmid |H|$ : Then $p \mid [K : H] < |K|$.

As $K$ is abelian, $K' := K/H$ is a group. But $|K'| < |K|$.
So by the inductive hypothesis $\exists\ a \in K'$ of order $p$.

i.e. $a = gH$, for some $g \in K$. Let $m := \mathrm{ord}(g)$.
So

$$a^m = (gH)^m = g^m H = H = 1_{K/H}$$

Therefore $p \mid m$ and we obtain the element

$$x := g^{\frac{m}{p}} \in K$$

of order $p$ and the proof is complete. $\square$

<u>Remark</u> The above lemma holds for a not necessarily abelian, finite group $G$. Indeed, if we proceede by induction on $|G|$:

Case I $\quad p \mid |Z(G)|$ : As $Z(G)$ is abelian, there exists $H \leqslant Z(G)$ s.t. $|H| = p$ and we are done.

Case II $\quad p \nmid |Z(G)|$ : But $p \mid |G|$. So the class eq'n implies that $\exists\, x \in C'$ s.t.

$$p \nmid [G : G_x]$$

Therefore $p \mid |G_x| < |G|$. By the inductive hypothesis $\exists\, H < G_x$ s.t. $|H| = p$ and we are done.