Lecture 4 Sylow's theorems

Thm (1st Sylow) Let G be a finite group and suppose a prime $p \mid |G|$. Then $\exists$ p-Sylow subgroup of G.

Proof — by induction on $|G|$

WLOG we may assume $|G| \neq p$. If $H \lneq G$ & $p \nmid [G:H]$, as $p \mid |G| = |H| \cdot [G:H]$, we have $p^n \| |H|$.

So WLOG we may assume that $p \mid [G:H]$ for $H \lneq G$.

The class formula

$$|G| = |Z(G)| + \sum_{g \in C'} [G : G_g]$$

implies that $p \mid |Z(G)|$. Thus $Z(G) \neq \{e\}$.

Then $\exists\ H \leq Z(G)$ s.t. $|H| = p$. As $Z(G)$ is the centre of G, we see that $H \trianglelefteq G$ and

$$p^{n-1} \,\|\, G/H$$

where $p^n \,\|\, |G|$. By induction $\exists\ K' \leq G/H$ s.t. $|K'| = p^{n-1}$

We have $H \trianglelefteq K$ and

$$
\begin{tikzcd}
 & & 1 \arrow[d, dash] & 1 \arrow[d, dash] \\
 & p & H \arrow[r, dash] \arrow[d, dash] & H \arrow[d, dash] \\
\Rightarrow & p^n & \pi^{-1}(K') =: K \arrow[r, dash] \arrow[d, dash] & G \arrow[d, "\pi"] \\
 & p^{n-1} & K/H = K' \arrow[r, dash] \arrow[d] & G/H \arrow[d] \\
 & & 1 & 1
\end{tikzcd}
$$

$\square$

<u>Def'n</u> Given a group G and $H \leq G$, the normalizer of H,

$$N_G(H) := \{ x \in G : \quad xHx^{-1} = H \} \trianglerighteq H$$

It is the largest $H' \leq G$ s.t. $H' \trianglerighteq H$.

We have $N_G(H) = G_H$, the isotropy group of G acting on

$$\mathcal{G} := \{ H \leq G \}$$

by conjugation.

Thm (Sylow) Pick a prime $p \mid |G| < \infty$. Then for every p-subgroup $H \leqslant G$ $\exists$ p-Sylow supergroup P'.

Proof

Let G act on $\mathcal{G}$ by conjugation. Pick a p-Sylow subgroup $P \leqslant G$ and put

$$\mathcal{P} := G \cdot P.$$

Claim We have $p \nmid |P|$.

Proof

Recall that $|G \cdot P| = [G : G_P]$, with $G_P = N_G(P)$.

But the chain $G \geq G_P \geq P$ yields

$$|G| = [G : G_P]\, |G_P| \quad \& \quad |G_P| = [G_P : P]\, |P|$$

So $|G| = [G : G_P][G_P : P]\,|P|$ and thus $p \nmid |G \cdot P|$. $\square$

<u>Claim</u> $\exists$ p-Sylow P' s.t. $H \leq N_G(P')$.

Proof

Let $H \leq G$ be a p-subgroup. It acts on $\mathcal{P}$ so

$$p \nmid |\mathcal{P}| = \sum_{i=1}^{r} |H \cdot P_i| = \sum_{i=1}^{r} [H : H_{P_i}].$$

But $p^n = |H| = [H : H_{P_i}]\,|H_{P_i}|$; if $H_{P_i} \lneq H$

then $p \mid [H : H_{P_i}]$. Therefore $\exists$ $i_0 \in \{1, \dots, r\}$

s.t. $P' := P_{i_0}$ $\quad |H \cdot P'| = 1$.

Therefore $H \leq G_{P'} = N_G(P')$ $\square$

<u>Claim</u> $P' \trianglelefteq P'H \leq G$ and

$P'H/P'$

$$\begin{tikzcd} & P'H & \\ P' \arrow[ur, dash] & & H \arrow[ul, dash] \\ & P' \cap H \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

$H/P' \cap H$

Proof

[Ex.]

Hence $|P'H/P'| = p^{\alpha}$, so $|P'H| = p^{\beta}$ and obviously $P'H \geqslant P'$.

But P' is a maximal p-subgroup of G, so

$$P'H = P'.$$

Therefore $H \subseteq P'$ $\square$