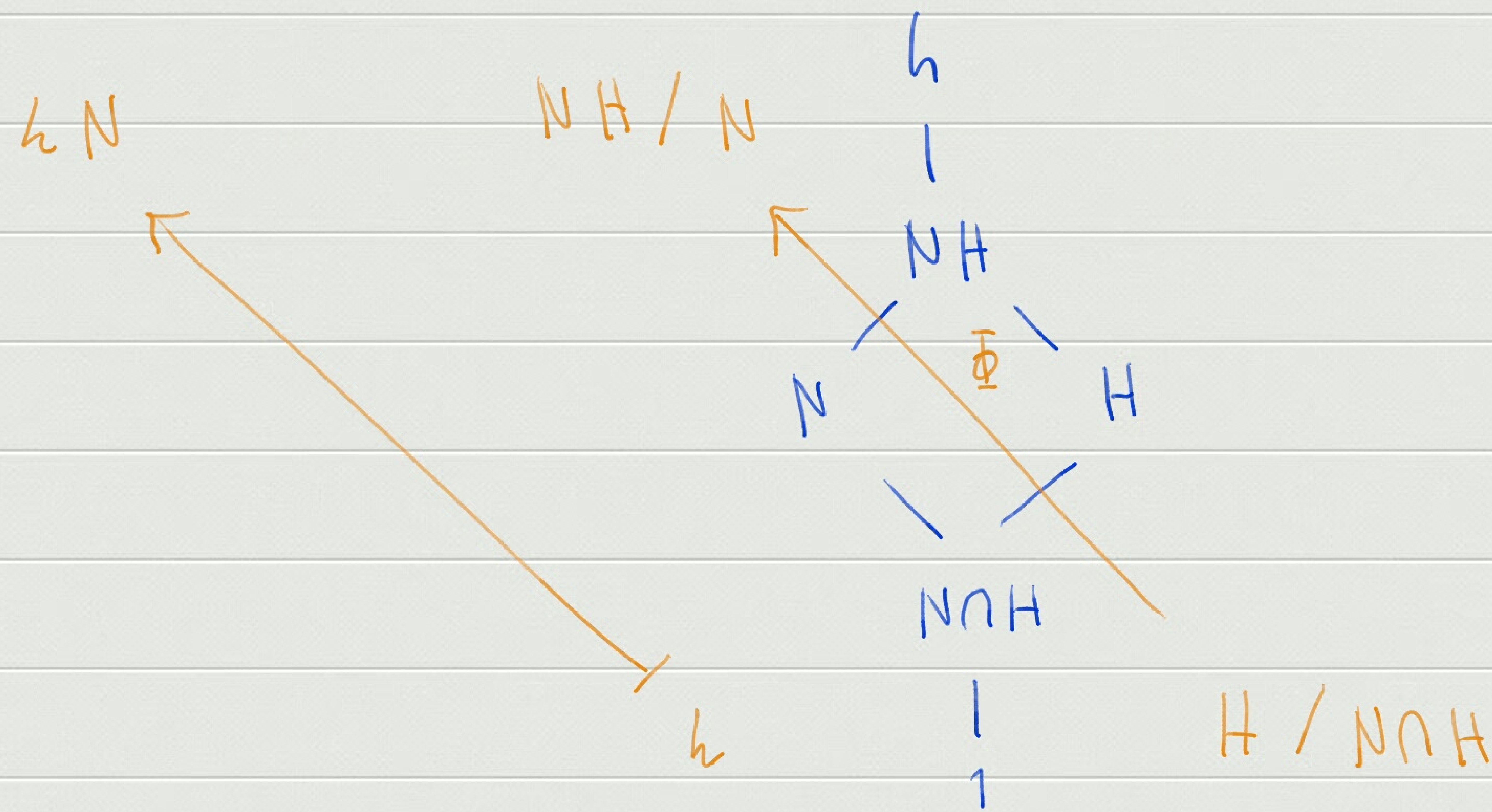
Lecture 5 Preliminaries

Recall that the second isomorphism says that given a group G and subgroups $H \leq G$ and $N \trianglelefteq G$, we have



In particular, if $\underbrace{G = NH \text{ and } 1 = N \cap H}_{\star}$ then we have an isomorphism

$$\Phi : H \longrightarrow G/N$$

This motivates the following

<u>Defn</u> If ($\star$) then we write

$$G = N \ltimes H$$

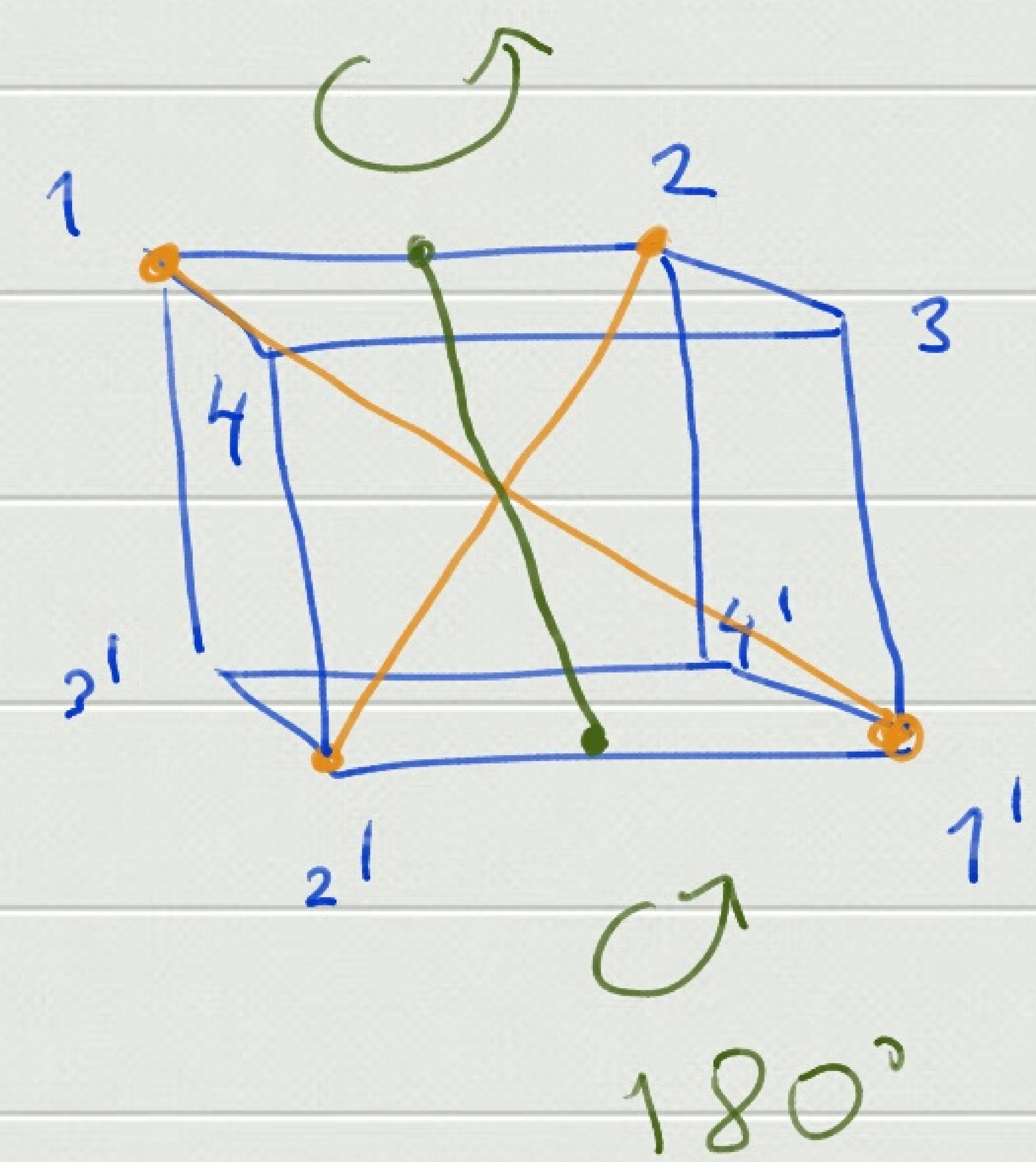
and say that G is the *semidirect product* of N and H.

Example We may regard the symmetric group S_4 inside $SO(3)$ via the group homomorphism

$$\rho : S_4 \longrightarrow SO(3)$$

by extending the action of S_4 on the 4 main diagonals of the hexahedron to rotations of $\mathbb{R}^3$, e.g.

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \longmapsto$$



In fact, if $B := \{\tau_1, \tau_2, \tau_3\}$ is the set of adjacent transpositions

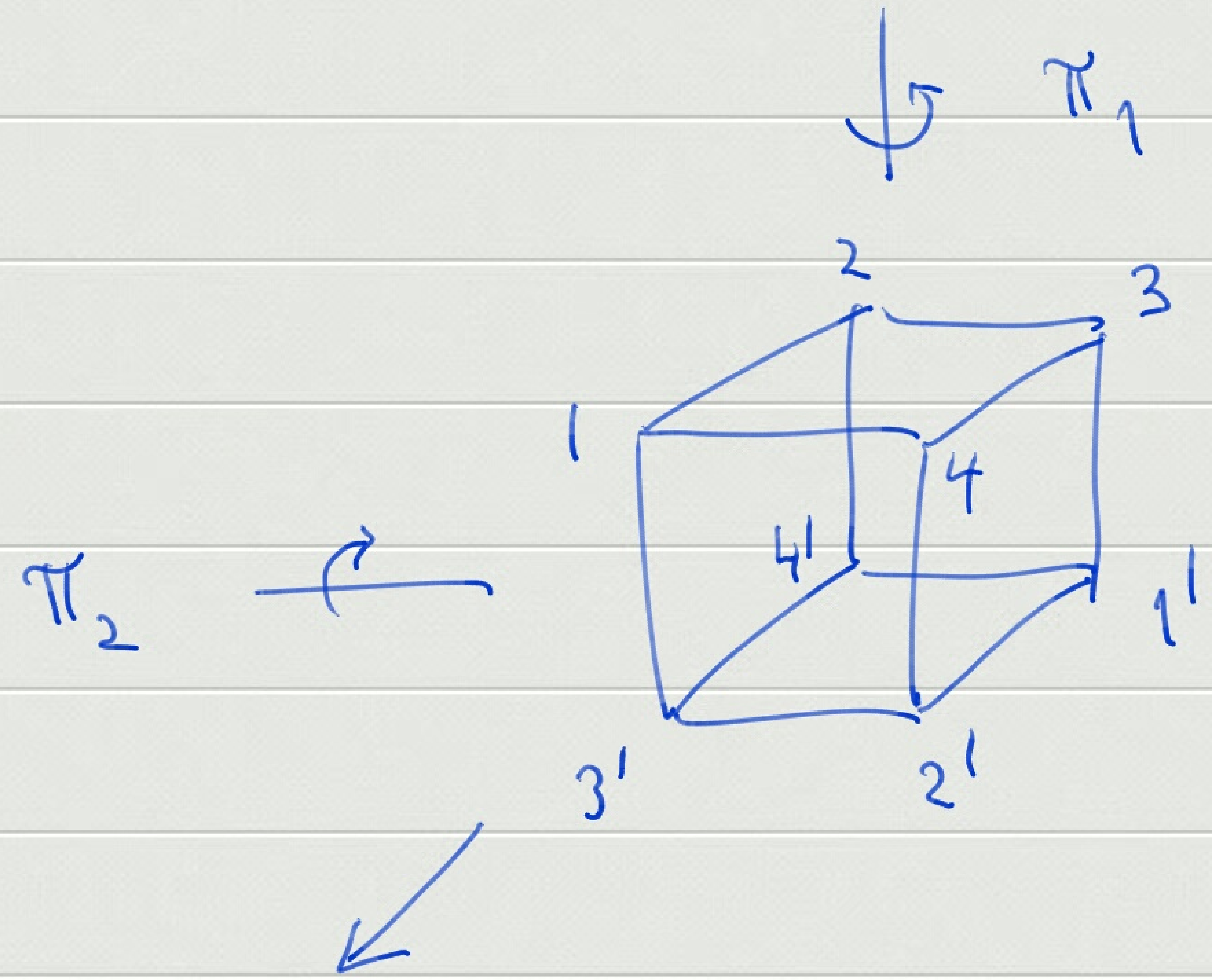
$$\tau_1 := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \quad \tau_2 =: \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \quad \tau_3 := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

then

$$S_4 = \langle B \rangle$$

and the attached Cayley graph $\mathcal{C}_B$ turns out to be the Permutohedron. Using $\mathcal{C}_B$ we get

$$\pi_1 := \tau_1 \tau_2 \tau_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \quad \pi_2 := \tau_1 \pi_1 \tau_1 = \tau_2 \tau_3 \tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$



$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = \tau_3 \tau_2 \tau_3 \tau_1 \tau_2$$

We have

$$\rho(\tau_1) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\rho(\tau_2) = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\rho(\tau_3) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Therefore

$$\rho(\pi_1) = \rho(\tau_1)\,\rho(\tau_2)\,\rho(\tau_3) = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix},$$

with eigenspaces generated by

$$\begin{array}{cccc} & \lambda = 1 & \lambda = -i & \lambda = i \\ e_1 := & \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, & \begin{bmatrix} i \\ 1+i \\ 1 \end{bmatrix}, & \begin{bmatrix} -i \\ 1-i \\ 1 \end{bmatrix} \end{array} \qquad u_1 := \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Similarly,

$$\rho(\pi_2) = \rho(\tau_2)\rho(\tau_3)\rho(\tau_1) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix},$$ with eigenspaces

generated by

$$\beta_2 := \underset{\lambda = 1}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}, \underset{\lambda = -i}{\begin{bmatrix} 1 \\ i+1 \\ 1 \end{bmatrix}}, \underset{\lambda = i}{\begin{bmatrix} 1 \\ -i+1 \\ 1 \end{bmatrix}}$$

$$V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Finally

$$\rho(\pi_3) = \rho(\tau_3)\rho(\tau_2)\rho(\tau_3)\rho(\tau_1)\rho(\tau_2) = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

with eigenspaces generated by

$$\begin{array}{cccc} & \lambda = 1 & \lambda = i & \lambda = -i \\ \beta_3 := & \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, & \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}, & \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \end{array}$$

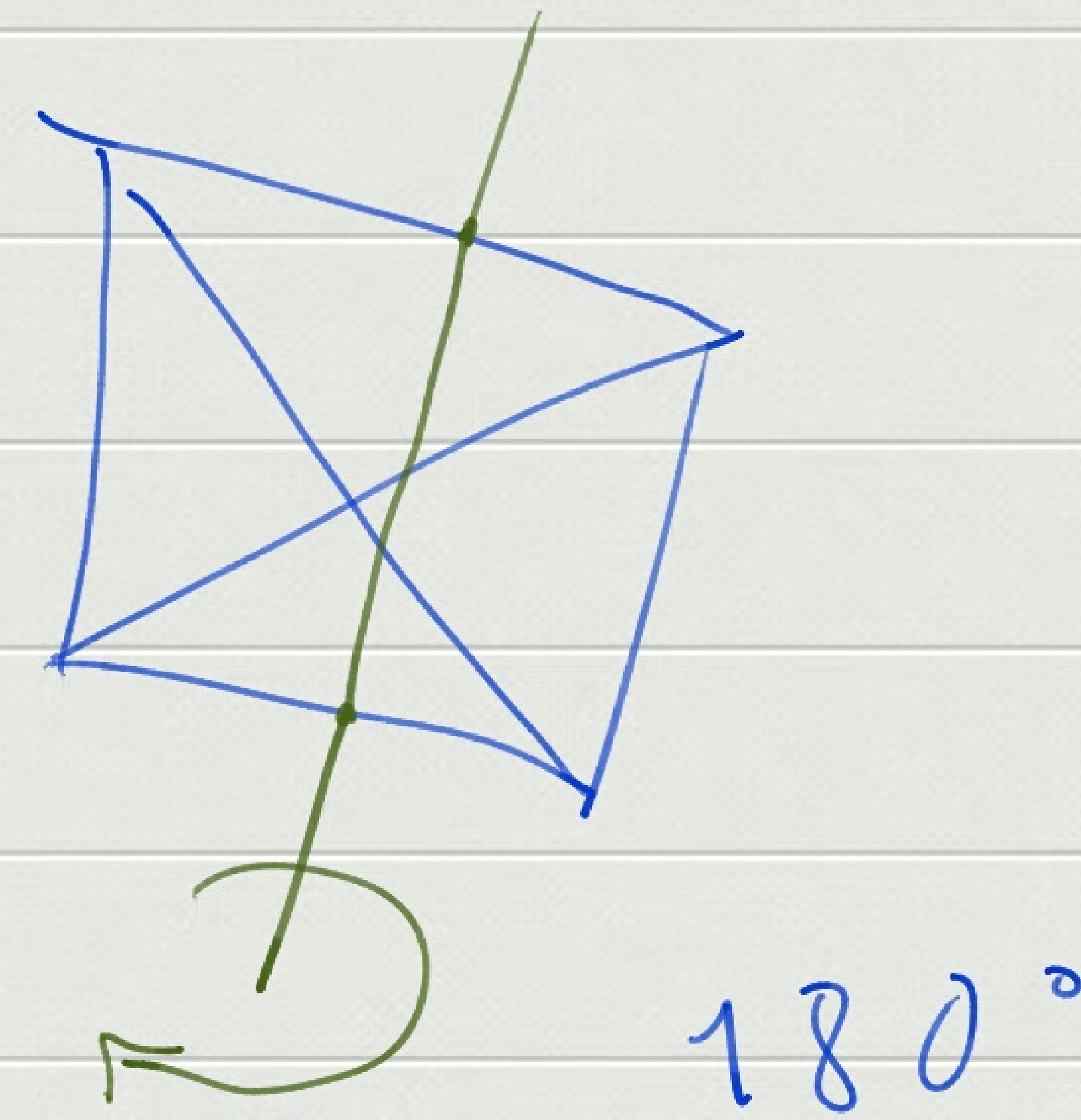
$$w_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Consider that we may embed the tetrahedron T inside the hexahedron

The subgroup of S_4 that maps T to itself is the alternating group A_4 and

$$G = A_4 \ltimes H,$$

where $H := \langle \tau_i \rangle$. Moreover, We have $A_4 = \mathbb{V} \rtimes K$, where $\mathbb{V}$ is Klein's Vierergruppe



(all rotations of $180°$ WRT edges)

and $K = \langle r \rangle$, where r is any rotation of $60°$ WRT a vertex of the tetrahedron.

We thus have a chain of the form

$$
\begin{array}{ll}
S_4 & \\
\trianglerighteq & S_4 / A_4 \cong C_2 \\
A_4 & \\
\trianglerighteq & A_4 / \mathbb{V} \cong C_3 \\
\mathbb{V} & \\
\trianglerighteq & \mathbb{V} / C_2 \cong C_2 \\
C_2 & \\
\trianglerighteq & C_2 / 1 \cong C_2 \\
1 &
\end{array}
$$

Def'n We say that a finite group is solvable if

$$
\begin{array}{lll}
G & & \\
\triangledown\!| & G/G_1 & \text{cyclic} \\
G_1 & & \\
\triangledown\!| & G_1/G_2 & \text{cyclic} \\
G_2 & & \\
\triangledown\!| & \vdots & \vdots \\
\vdots & & \\
\triangledown\!| & & \\
G_k = 1 & &
\end{array}
$$

Classification of p-groups up to p^3

I. $|G| = p$: $G \cong C_p$

II. $|G| = p^2$: either $G \cong C_{p^2}$ or $G \cong C_p \times C_p =: C_p^2$

III. $|G| = p^3$: C_{p^3}, $C_{p^2} \times C_p$, C_p^3, which are abelian and

for $p > 2$, either

the Heisenberg group of $\mathbb{F}_p$

$$G \cong C_p^2 \rtimes C_p \cong \mathcal{H}(\mathbb{F}_p) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{F}_p \right\}$$

or

$G \cong C_{p^2} \rtimes C_p$ (a non-abelian nilpotent group)

and for $p = 2$, either $G \cong D_4$ or $G \cong Q_8$.