Lecture 6 The symmetric group S_n and the alternating group A_n

We may define the alternating group A_n as the subgroup of S_n that fixes the polynomial

$$f(X_1, \dots, X_n) := \prod_{1 \leq i < j \leq n} (X_i - X_j),$$

where S_n acts on f by permuting its n variables

$$f(X_1, \dots, X_n) \longmapsto f(X_{\pi(1)}, \dots, X_{\pi(n)}),$$

$\forall \pi \in S_n$. More precisely, we have

$$\delta(X_1, \dots, X_n) = \sigma(\pi)\, f(X_{\pi(1)}, \dots, X_{\pi(n)})$$

where $\sigma(\pi) = \pm 1$ and we have a group homomorphism

$$\begin{aligned} S_n &\longrightarrow \{-1, 1\} \\ \pi &\longmapsto \sigma(\pi) \end{aligned}$$

whose kernel is A_n. In particular, $A_n \trianglelefteq S_n$.

The above assertions follow easily by induction and are left as exercises for the reader.

We'll now define the partition $(\lambda_1, \lambda_2, \dots, \lambda_r)$ of n,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_r, \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r,$$

naturally attached to any given $\pi \in S_n$ by inductively picking

$x_1, x_2, \dots \in \{1, 2, \dots, n\}$,

$$* \begin{cases} \pi x_1, \pi^2 x_1, \dots, \pi^{\lambda_1} x_1 = x_1 \\ \pi x_2, \pi^2 x_2, \dots, \pi^{\lambda_2} x_2 = x_2 \\ \quad\vdots \qquad\qquad\qquad\quad \vdots \\ \pi x_r, \pi^2 x_r, \dots, \pi^{\lambda_r} x_r = x_r \end{cases}$$

until the above rows list exactly once the integers $1, 2, \dots, n$, and then reorder the rows $(*)$ thus obtained so that

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_r$$

We shall express this by writing

$$\pi \vdash (\lambda_1, \lambda_2, \dots, \lambda_r).$$

<u>Propn</u> Given π, $\sigma \in S_n$. The following are equivalent

(i) $\exists\ \kappa \in S_n$ s.t. $\sigma = \kappa \pi \kappa^{-1}$

(ii) $\pi, \sigma \vdash (\lambda_1, \dots, \lambda_r)$

Proof

Suppose $\sigma = \kappa \pi \kappa^{-1}$, where $\sigma, \kappa, \pi \in S_n$. Then

$$\begin{matrix} \sigma x_1', & \sigma^2 x_1', & \ldots, & \sigma^{\lambda_1} x_1' \\ \vdots & \vdots & & \vdots \\ \sigma x_r', & \sigma^2 x_r', & \ldots, & \sigma^{\lambda_r} x_r' \end{matrix}$$

where $x_i' := \kappa x_i$, for each $i \in \{1, \ldots, r\}$. The converse is clear $\square$

Defi: Given $\pi \in S_n$ its cycles expression is

$$\pi = (\pi x_1 \cdots \pi^{\lambda_1} x_1)(\pi x_2 \cdots \pi^{\lambda_2} x_2) \cdots (\pi x_r \cdots \pi^{\lambda_r} x_r),$$

where we agree to remove the cycles $(\cdot)$ of length 1.

From the above proof we may see that $\forall\, \kappa \in S_n$ we have the formula

$$\kappa \underbrace{(i_{11} \cdots i_{1\lambda_1}) \cdots (i_{r1} \cdots i_{r\lambda_r})}_{=\,\pi} \kappa^{-1} = (\kappa i_{11} \cdots \kappa i_{1\lambda_1}) \cdots (\kappa i_{r1} \cdots \kappa i_{r\lambda_r})$$

We may interpret the above in terms of the language of the preceeding section as follows. Consider the rep'n

$$H \xrightarrow{S_{\mid H}} \mathrm{Perm}(\{1, \ldots, n\}) = S_n$$

where $H = \pi^{\mathbb{Z}} = \{\ldots, \pi^{-1}, \pi^0, \pi^1, \pi^2, \ldots\}$.

Then the rows $(*)$ give the orbit decomposition of $\{1, \ldots, n\}$ with respect to the action of $H = \pi^{\mathbb{Z}}$. This orbit decomposition is known as the *cycle decomposition* attached to π.

Prop^n Every $\pi \in S_n$ may be decomposed as

$$\pi = \tau_1 \tau_2 \cdots \tau_k ,$$

where $\tau_1, \tau_2, \dots, \tau_k$ are transpositions.

Proof

Just apply the Bubble Sort algorithm $\square$

Corollary We have

$$A_n = \{ \tau_1 \tau_2 \cdots \tau_k \mid \tau_i \text{ transp}, \ k \equiv 0 \pmod 2 \}.$$