

Lecture 7 Simplicity of A_n for $n \geq 5$

Lemma 1 If $n \geq 3$ then $A_n = \langle \text{3-cycles in } A_n \rangle$.

Proof

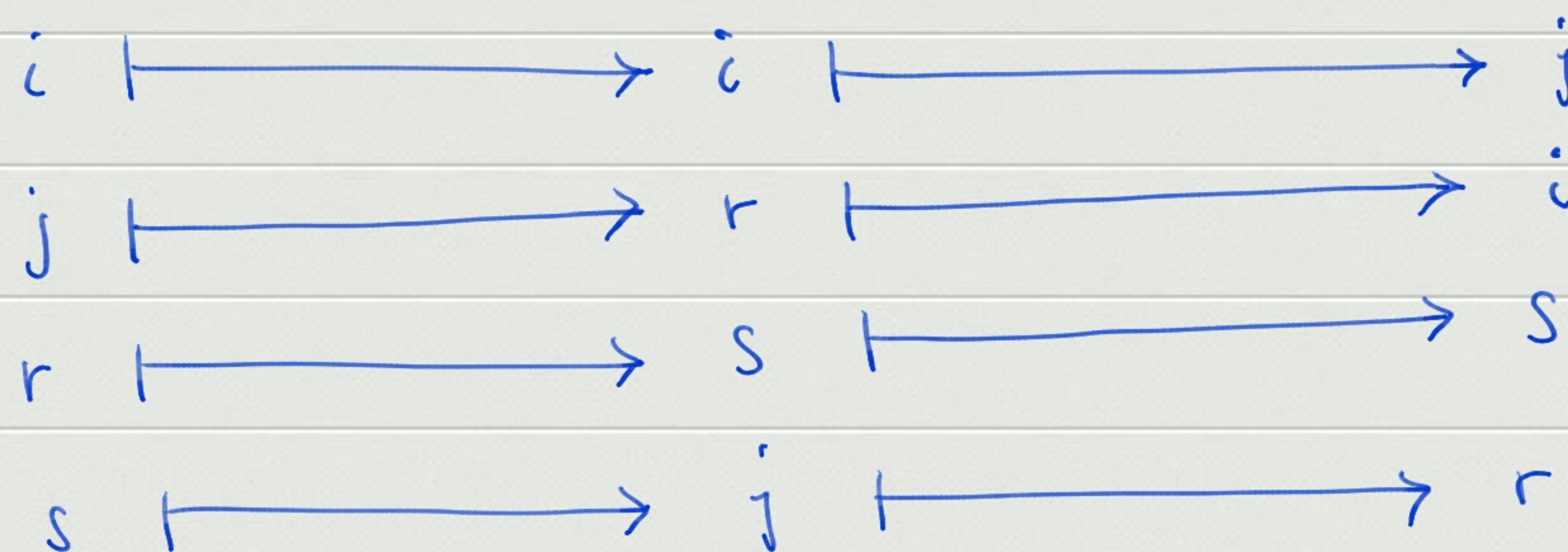
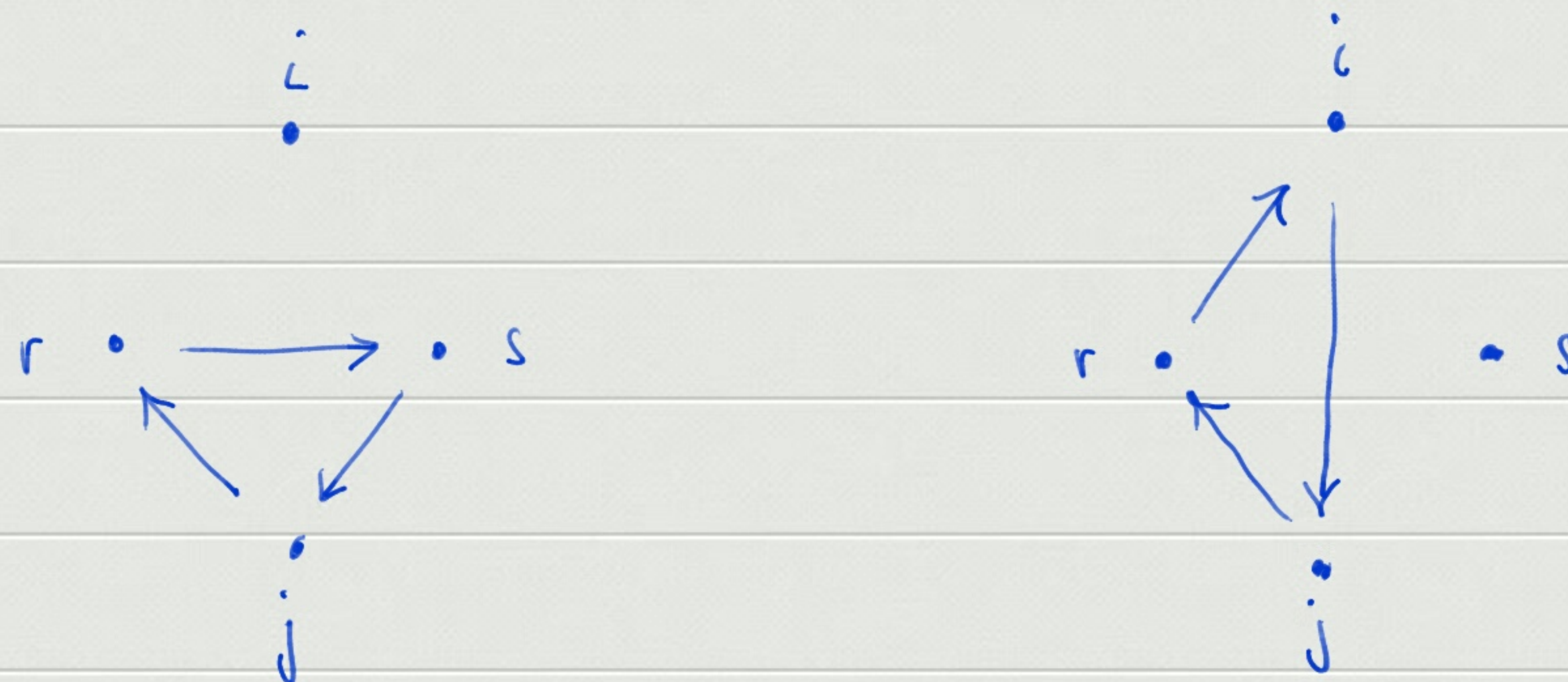
Consider the product of two transpositions $(ij)(rs)$

Case 1 They have an element in common. Then this product is either the identity or a 3-cycle.

Case 2 They have no element in common. Then

$$(ij)(rs) = (ijr)(jrs)$$

Indeed, just consider that



So the lemma follows from the above Corollary \square

Lemma 2 Let $n \geq 3$ and suppose that we have $N \leq A_n$.

If N contains a 3-cycle then $N = A_n$.

Proof

Fix $i, j \in \{1, \dots, n\}$ and let

$$S_{i,j} := \{ (ija) \mid a \neq i \text{ \& } a \neq j \}.$$

Claim We have $A_n = \langle S_{ij} \rangle$.

Proof of claim

By lemma 1 it suffices to show that every 3-cycle $(abc) \in \langle S_{i,j} \rangle$. But this follows from

$$(abc) = (ija)^{-1} (ijc) (ijb)^{-1} (ija) \quad \square \text{ (claim)}$$

Now suppose that some $g^{-1} := (ija) \in S_{ij}$ lives in N and that $N \trianglelefteq A_n$. Put $k := (ij)(ak)$, where $k \in \{1, \dots, n\}$, so

$$k g k^{-1} = (ijk) \in N.$$

Therefore $N = A_n$ \square

Lemma 3 If $n \geq 5$ and $N \trianglelefteq A_n$ then N contains a 3-cycle.

Proof

Let $1 \neq N \trianglelefteq A_n$ and pick $\sigma \in N$ s.t.

$$\sigma = (i_{11} \cdots i_{1\lambda_1}) \cdots (i_{r1} \cdots i_{r\lambda_r}) \quad \star$$

has minimal length. It will suffice to show that either $r = 1$ & $\lambda_1 = 3$ so

$$\sigma = (i_{11} i_{12} i_{13})$$

Consider the following.

Claim We may not have $\lambda_1 = \dots = \lambda_r = 2$.

Proof of claim

Since $\sigma \in N \subseteq A_n$, then $r \geq 2$. Therefore (A) is of the form

$$\dots (ij)(rs) \dots$$

As $n \geq 5$, $\exists k \in \{1, \dots, n\}$ s.t. $k \neq i, j, r, s$. Now let

$$\tau := (rsk)$$

and consider $\sigma' := [\tau, \sigma] = \tau \sigma \tau^{-1} \sigma^{-1} \in N$.

Note that σ' has more fixed points than σ (it has i and j as extra fixed points), a contradiction \square (claim)

So we may assume that

$$\exists k \in \{1, \dots, r\} \text{ s.t. } \lambda_k \geq 3$$

Then (A) is of the form

$$\dots (i j k \dots) \dots$$

If σ is a 3-cycle then we are done. Otherwise, as σ is even, it moves two other elements $r, s \in \{1, \dots, n\}$. (Simply note that $(i j k r)$ is an odd permutation.) Let $\tau := (k r s)$ and as above $\tau' := [\tau, \sigma] \in N$ and again σ' has an extra fixed point (i.e. $i \xrightarrow{\sigma'} i$), a contradiction \square

Thm The group A_n is simple if $n \geq 5$.

Proof

Suppose $N \triangleleft A_n$. By Lemma 3, N contains a 3-cycle.

Then Lemma 2 implies that $N = A_n$ and the thm follows \square