Lecture 7 Simplicity of A_n for $n \geq 5$

Lemma 1 If $n \geq 3$ then $A_n = \langle$ 3-cycles in $A_n \rangle$.

Proof

Consider the product of two transpositions $(ij)(rs)$

Case 1 They have an element in common. Then this product is either the identity or a 3-cycle.

Case 2 They have no element in common. Then

$$(ij)(rs) = (ijr)(jrs)$$

Indeed, just consider that

$$
\begin{tikzcd}
 & i & \\
r \arrow[rr] & & s \arrow[dl] \\
 & j \arrow[ul] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & i \arrow[dd] & \\
r \arrow[ur] & & s \\
 & j \arrow[ul] &
\end{tikzcd}
$$

$$
\begin{array}{ccccc}
i & \longmapsto & i & \longmapsto & j \\
j & \longmapsto & r & \longmapsto & i \\
r & \longmapsto & s & \longmapsto & s \\
s & \longmapsto & j & \longmapsto & r
\end{array}
$$

So the lemma follows from the above Corollary $\square$

Lemma 2 Let $n \geqslant 3$ and suppose that we have $N \trianglelefteq A_n$. If N contains a 3-cycle then $N = A_n$.

Proof

Fix $i, j \in \{1, \dots, n\}$ and let

$$S_{i,j} := \{ (i\,j\,a) \mid a \neq i \ \& \ a \neq j \}.$$

Claim We have $A_n = \langle S_{ij} \rangle$.

Proof of claim

By Lemma 1 it suffices to show that every 3-cycle $(a\,b\,c) \in \langle S_{i,j} \rangle$. But this follows from

$$(a\,b\,c) = (i\,j\,a)^{-1}(i\,j\,c)(i\,j\,b)^{-1}(i\,j\,a)$$

$\square$ Claim

Now suppose that some $g^{-1} := (i\,j\,a) \in S_{ij}$ lives in N and that $N \trianglelefteq A_n$. Put $\kappa := (i\,j)(a\,k)$, where $k \in \{1, \dots, n\}$, so

$$\kappa g \kappa^{-1} = (i\,j\,k) \in N.$$

Therefore $N = A_n$ $\square$

Lemma 3 If $n \geqslant 5$ and $N \trianglelefteq A_n$ then N contains a 3-cycle.

Proof

Let $1 \ntrianglelefteq N \ntrianglelefteq A_n$ and pick $\sigma \in N$ s.t.

$$\sigma = (i_{11} \cdots i_{1\lambda_1}) \cdots (i_{r1} \cdots i_{\lambda_r}) \qquad \star$$

has minimal length. It will suffice to show that either $r = 1$ & $\lambda_1 = 3$ so

$$\sigma = (i_{11}\ i_{12}\ i_{13})$$

Consider the following.

Claim We may not have $\lambda_1 = \dots = \lambda_r = 2$.

Proof of claim

Since $\sigma \in N \subseteq A_n$, then $r \geq 2$. Therefore $(\bigstar)$ is of the form

$$\dots (ij)(rs) \dots$$

As $n \geq 5$, $\exists\, k \in \{1, \dots, n\}$ s.t. $k \neq i, j, r, s$. Now let

$$\tau := (r\,s\,k)$$

and consider $\sigma' := [\tau, \sigma] = \tau \sigma \tau^{-1} \sigma^{-1} \in N$.

Note that σ' has <u>more</u> fixed points than σ (it has i and j as extra fixed points), a contradiction $\square$ Claim

So we may assume that

$$\exists\ k \in \{1, \dots, r\} \text{ s.t. } \lambda_k \geq 3$$

Then $(\cancel{A})$ is of the form

$$\dots (i\ j\ k \dots) \dots$$

If σ is a 3-cycle then we are done. Otherwise, as σ is even, it moves two other elements $r, s \in \{1, \dots, n\}$. (Simply note that $(i\,j\,k\,r)$ is an odd permutation.) Let $\tau := (k\,r\,s)$ and as above $\sigma' := [\tau, \sigma] \in N$ and again σ' has an extra fixed point (i.e. $i \xmapsto{\sigma'} i$), a contradiction $\square$

Thm The group A_n is simple if $n \geqslant 5$.

Proof

Suppose $N \trianglelefteq A_n$. By Lemma 3, N contains a 3-cycle. Then Lemma 2 implies that $N = A_n$ and the thm follows $\square$