Lecture 9 Simplicity of $M_n(D)$, D a division ring

Fix $n \in \{1, 2, \dots\}$ and a ring S, assumed associative and with $1_S \in S$. Consider the ring $M_n(S)$ of n by n matrices with entries in S. For each $k, i \in \{1, \dots, n\}$ let

$$e_{ki} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \underset{\substack{\uparrow \\ \text{column } i}}{\cdots} & 0 \end{bmatrix} \leftarrow \text{row } k.$$

We have $\forall\ i, j, k, l \in \{1, \dots n\}$:

$e_{ki} e_{jl} = \delta_{ij} e_{kl}$, and $\forall A \in M_n(S)$

$$e_{ki} A = \begin{array}{cl} \begin{bmatrix} 0 \; 0 & \cdots & 0 \; 0 \\ & i\text{th row of } A & \\ 0 \; 0 & \cdots & 0 \; 0 \end{bmatrix} & \begin{array}{l} \\ \leftarrow \text{row } k \\ \\ \end{array} \end{array},$$

so

$$e_{ki} A e_{jl} = \begin{array}{cl} \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & a_{ij} & \vdots \\ 0 & \cdots & \cdot & 0 \end{bmatrix} & \begin{array}{l} \\ \leftarrow \text{row } k \\ \\ \end{array} \\ \begin{array}{c} \uparrow \\ \text{column } l \end{array} & \end{array}$$

$$= a_{ij} e_{kl}$$

Prop^n Let S be a ring. Then

(a) $I \trianglelefteq S \Rightarrow M_2(I) \trianglelefteq M_2(S)$

(b) $J \trianglelefteq M_2(S) \Rightarrow \exists\, I \trianglelefteq S$ s.t. $J = M_2(I)$.

Proof

(a) Easy — an exercise for the reader.

(b) If $J \trianglelefteq M_2(S)$ then we put $I := \{ a = A_{11} : A \in J \}$.

Claim Then $I \trianglelefteq S$.

Proof of claim

If $a, b \in I$ then $\exists\ A, B \in J$ s.t. $a = A_{11}$ & $b = B_{11}$, so

$$J \ni e_{11} A e_{11} = a e_{11} \quad \& \quad J \ni e_{11} B e_{11} = b e_{11}.$$

Hence $(a+b) e_{11} \in J$ and $a+b \in I$. We also have

$$J \ni (r e_{11})(a e_{11}) = r a e_{11},$$

$\forall r \in S$, so $ra \in I$. Similarly,

$$J \ni (a e_{11})(r e_{11}) = a r e_{11},$$

$\forall r \in S$, so $ar \in I$. Therefore $I \trianglelefteq S$ $\square$

If $A = \begin{pmatrix} a_{11} & \cdot & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdot & a_{nn} \end{pmatrix} \in J$, then $\forall\, i,j \in \{1,\dots,n\}$:

$$J \ni e_{1i} A e_{j1} = a_{ij} e_{11},$$

so $a_{ij} \in I$. Hence $J \subseteq M_n(I)$. Conversely, pick any $A = \begin{pmatrix} a_{11} & \cdot & a_{1n} \\ \cdot & & \cdot \\ a_{n1} & \cdot & a_{nn} \end{pmatrix} \in M_n(I)$. So $\forall\, k,l \in \{1,\dots,n\}$ there $\exists\, B = \begin{pmatrix} b_{11} & \cdot & b_{1n} \\ \cdot & & \cdot \\ b_{n1} & \cdot & b_{nn} \end{pmatrix} \in J$ s.t. $a_{kl} = b_{11}$. But $J \ni e_{k1} B e_{1l} = b_{11} e_{kl} = a_{kl} e_{kl}$. Thus $A = \sum_{kl} a_{kl} e_{kl} \in J$ $\square$

Defn We say a ring R is *simple* if $\forall\, I \trianglelefteq R$ either $I = R$ or $I = \{0\}$.

Recall that a division ring D is a ring s.t. its group of units

$$D^\times = D - \{0\}.$$

In particular, D is simple. So the above proposition implies that $\forall\, n \in \{1, 2, 3, \dots\}$ the ring $M_n(D)$ is simple.