Lecture 10 Some left and right module structures

<u>Defn</u> Given a ring D, a right D-module is an abelian group $(M, +)$ together with a map

$$M \times D \longrightarrow M$$
$$(x, a) \longmapsto x \cdot a$$

s.t. $\forall\, x, y \in M$ and $a, b \in D$:

$(x \cdot a) \cdot b = x \cdot (ab)$, $\quad x \cdot 1_D = x$;

$(x + y) \cdot a = x \cdot a + y \cdot b$, $\quad x \cdot (a+b) = x \cdot a + x \cdot b$.

We'll denote M_D a right D-module. Given right D-modules M_D and N_D, a right D-module homomorphism is a map $M_D \xrightarrow{f} N_D$ s.t. $\forall\, m_1, m_2 \in M_D$ and $\forall\, a, b \in D$:

$$f(m_1 \cdot a + m_2 \cdot b) = f(m_1) \cdot a + f(m_2)\, b \qquad (\star)$$

We'll denote $\mathrm{End}(M_D)$ the set of right D-linear maps $M_D \xrightarrow{f} M_D$. Component-wise addition and composition of maps turn $\mathrm{End}(M_D)$ into a ring.

Note that M_D, above, has also a natural structure of a left R-module, where $R := \mathrm{End}(M_D)$. Indeed, we may define

$$\begin{aligned} R \times M_D &\longrightarrow M \\ (r, m) &\longmapsto r \cdot m := r(m) \end{aligned}$$

In particular, Eq'n ($\star$) says that $\forall\ m \in M_D,\ r \in R,\ a \in D$:

$r(m \cdot a) = r(m) \cdot a$, which we may state as

$$r \cdot (m \cdot a) = (r \cdot m) \cdot a \qquad *^1$$

Property $(*^1)$ says that M is a (D, R)-bimodule.

Prop'n 2 We have a direct sum decomposition $R := M_n(D) = \bigoplus_{i=1}^{n} \mathfrak{a}_i$,

where for $i \in \{1, \dots, n\}$

$$\mathfrak{a}_i := R e_{ii} = \left\{ A e_{ii} = \begin{bmatrix} 0 & \cdots & 0 & a_{1i} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_{2i} & 0 & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \cdots & 0 & a_{ni} & 0 & \cdots & 0 \end{bmatrix} \;\middle|\; A \in R \right\}$$

is a simple left R-module and a right $e_{ii} R e_{ii} =: D_i$-module

with $1_{D_i} = e_{ii}$ and $D_i \cong D$ s.t. $\forall\, r \in R,\ x \in D_i,\ m \in \mathfrak{a}_i$

$$(r \cdot m) \cdot x = r \cdot (m \cdot x). \qquad *^2$$

Remark If D is a division ring and M is a finitely generated left / right D-module, $\exists$ finite basis $\mathcal{B}$ for M over D, so

$$M \xrightarrow{\sim} D^n$$

$$a_1\beta_1 + \cdots a_n\beta_n = v \longmapsto (a_1, \ldots, a_n)$$

when $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ and thus

$$\mathrm{End}(M) \xrightarrow{\sim} M_n(D),$$

$$L \longmapsto [L]_{\mathcal{B}}$$

Indeed, the latter map is bijective and such that $[1]_{\mathcal{B}} = I_n$ and

$$[L_1 + L_2]_{\mathcal{B}} = [L_1]_{\mathcal{B}} + [L_2]_{\mathcal{B}}$$

$$[L_1 \circ L_2]_{\mathcal{B}} = [L_1]_{\mathcal{B}} [L_2]_{\mathcal{B}}$$

Thm (Wedderburn) If R is a simple ring that contains a simple left R-submodule then $\exists\, n \in \{1, 2, 3, \dots\}$ s.t.

$$R \cong M_n(D),$$

as rings, for some division ring D.

We'll prove this theorem by identifying in this context the objects predicted by Prop'n 2 and the Remark.