Lecture 11 Key lemma in our proof of Wedderburn's thm

Lemma Suppose $e \in R$ is an idempotent element s.t.

$$ReR = R. \qquad \star$$

Then

$$R \cong \mathrm{End}(M_D),$$

where

$$D = eRe, \qquad (1_D = e)$$

and

$$M = Re.$$

Ex Prove that $M = Re$ is both a left R-module and a right D-module, where $D = eRe$, and that moreover, we may see that $\forall\, r \in R,\ m \in M,\ x \in D$:

$$(r \cdot m) \cdot x = r \cdot (m \cdot x)$$

i.e M is an (R, D)-bimodule, so we may write ${}_R M_D$ to express this fact, or just M_D to stress the right D-module structure.

Proof of Lemma

Step 0: Define

$$R \xrightarrow{\rho} \mathrm{End}(M_D)$$

$$r \longmapsto \begin{pmatrix} M_D \xrightarrow{\rho_r} M_D \\ m \longmapsto rm \end{pmatrix}$$

This is a ring homomorphism.

<u>Step 1</u> (Injectivity of ρ): $\forall\, r \in \ker(\rho)$ we have

$$rRe = 0$$

$$\Downarrow$$

$$rReR = 0R = 0$$

$$\Downarrow \text{ by } (\ast)$$

$$rR = 0$$

But $1_R \in R$. Thus $r1_R = 0$, so $r = 0$. I.e. $\ker(\rho) = 0$.

Step 2 (Surjectivity of ρ): Pick any $f \in \mathrm{End}(M_D)$.

Note that ($\star$) implies that $\exists \{x_i\}_{i\in I}, \{y_i\}_{i\in I} \subseteq R$ s.t.

$$\sum_{i\in I} x_i e y_i = 1_R$$

Claim The ring homomorphism ρ maps $r_f := \sum_{i\in I} f(x_i e) e y_i \in R$ to f, i.e. $r_f m = f(m)$, $\forall m \in M$.

To prove our claim pick any $m \in M$ and compute

$$f(m) = f(1_R m) = f\Big(\Big[\sum_{i\in I} x_i e y_i\Big] m\Big) = \sum_{i\in I} f(x_i \underset{e^2}{\underset{\|}{e}}\, y_i \overset{me \;\; *}{\overset{\|}{m}}) =$$

$$\sum_{i\in I} f(x_i e \underbrace{e y_i m e}_{\in D}) = \sum_{i\in I} f(x_i e) e y_i m e$$

$$= \sum_{i\in I} f(x_i e) e y_i m = r_f m.$$

The claim follows and so does the lemma $\square$

* $m = re$ with $r \in R \Rightarrow me = re^2 = re \;\therefore\; m = me$.