# Lecture 12 Proof of Wedderburn's thm

**Claim** Suppose $\mathfrak{a} \subseteq R$ is a simple left $R$-module.
Then $\exists$ idempotent $e \in R$ s.t. $\mathfrak{a} = Re$, if $R$ is simple.

*Proof of claim*

We see that $\mathfrak{a}^2 \neq 0$. Indeed, $\exists\, m \in \mathfrak{a} - \{0\}$, so

$$(Rm)^2 = (RmR)m = Rm$$

So $\mathfrak{a}^2 = 0 \Rightarrow Rm = 0 \Rightarrow 1_R m = 0 \Rightarrow m = 0$ $\lightning$

So the non-trivial left $R$-module $\mathfrak{a}^2 \subseteq \mathfrak{a}$ has to be $\mathfrak{a}^2 = \mathfrak{a}$, as $\mathfrak{a}$ is a simple left $R$-module. In particular, there is $x \in \mathfrak{a} - \{0\}$ s.t. $\mathfrak{a} x \neq 0$. So $\mathfrak{a} x = \mathfrak{a}$ and $\exists\, \underset{\neq 0}{e} \in \mathfrak{a}$ s.t.

$$e x = x,$$

so

$$e^2 x = e x \quad \text{and} \quad (e^2 - e) x = 0.$$

The left $R$-module $\mathrm{Ann}_{\mathfrak{a}}(x) = \{ y \in \mathfrak{a} : y x = 0 \} \subseteq \mathfrak{a}$ is trivial. Indeed, $e \in \mathfrak{a}$ but $e \notin \mathrm{Ann}_{\mathfrak{a}}(x)$, as $e x = x \neq 0$; $\mathrm{Ann}_{\mathfrak{a}}(x) \subsetneq \mathfrak{a}$ and the simplicity of the left $R$-module $\Omega$ implies that $\mathrm{Ann}_{\Omega}(x) = 0$. But $e^2 - e \in \mathrm{Ann}_{\Omega}(x)$. Hence $e^2 - e = 0$, i.e. the idempotency condition $e^2 = e$. To complete the proof of our claim note that $Re$ is a nontrivial left $R$-submodule of $\Omega$. Indeed,

$$0 \neq e = e^2 \in Re \subseteq \Omega.$$

Therefore $Re = \Omega$ and our claim follows.

From the above claim and the lemma we have

$$R \cong \mathrm{End}(\Omega_D),$$

where $\Omega = Re$ and $e$ is as above and $D = eRe$.

<u>Claim</u> $D$ is a division ring.

*Proof of the claim*

Pick any non-zero $ere \in D$. Then

$$ere \in Rere \subseteq Re$$

But $Rere$ is a non-zero left $R$-submodule of $Re$.

Hence $Rere = Re$. So $\exists\, r' \in R$ s.t. $r'ere = e$, so

$$(er'e)(ere) = e,$$

i.e. every non-zero element in $D$ has a left inverse.

Therefore $\exists\, s \in D$ s.t.

$$e = s(ere)(er'e) = s(ere)e(er'e) =$$

$$\underbrace{s\,(ere)\,[(er'e)}_{=e}\,(ere)]\,(er'e) = (ere)(er'e),$$

every non-zero element of $D$ has a right inverse and the claim follows. To prove the thm it suffices to note that

$$0 \subsetneq I := \{x \in R : \dim_D \rho_x(Q) < \infty\} \trianglelefteq R,$$

as the simplicity of $R$ implies that $I = R$ $\square$