The Wedderburn-Artin Theorem

As proved by W.K. Nicholson

Preliminaries

- R will denote a non trivial associative ring with unity.
- If X, Y are additive subgroups of R, we define their product by

$$AA = \quad XY = \left\{ \sum_{i=1}^{n} x_i y_i \,\middle|\, n \geq 1,\ x_i \in X,\ y_i \in Y \right\}.$$

- We call R *semiprime* if $A^2 \neq 0$ for every nonzero ideal A in R.
- R is *simple* if it has no ideals other than 0 and R. Such a ring is necessarly semiprime.
- We say that R is *left artinian* if for every descending chain of left ideals $K_1 \supset K_2 \supset \cdots$, there is an n such that $K_n = K_{n+1} = \cdots$. This is equivalent to every nonempty family of left ideals having a minimal member.

Wedderburn-Artin Theorem

If R is a semiprime left artinian ring then.

$$R \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_r}(D_r)$$

where each D_i is a division ring and $M_n(D)$ denote the ring of $n \times n$ matrices over D.

Proof outline

We'll first prove that if R is simple with a minimal left ideal, then $R \cong M_n(D)$ (Wedderburn). Then we'll prove a key lemma that will allow us to reduce our theorem to reapeted uses of Wedderburn's Theorem.

Wedderbun Theorem

$0 \neq K \quad 0, K$

Brauer's Lemma.

Let K be a minimal left ideal of a ring R, such that $K^2 \neq 0$. Then $K = Re$ where $e^2 = e \in R$ and eRe is a division ring.

Proof.

Since $K^2 \neq 0$, certainly $Ku \neq 0$ for some $0 \neq u \in K$. Hence $Ku = K$ by minimality, so $eu = u$ for some $e \in K$.

$v \in K = Kv \quad ev = v$

$(re - r)v$

Now note that for $r \in K$, $re - r \in L = \{a \in K \mid au = 0\}$; since $L \subset K$ is a left ideal and $L \neq K$, it follows that $L = 0$ and $e^2 = e$. Thus $e \in Re \subset RK \subset K$, so by minimality $Re = K$.

$re = r$

$e^2 = e$

Wedderburn Theorem

Brauer's Lemma.

$e^2 = e$

e^2

Let $0 \neq b \in eRe$, then $0 \neq eb \in Rb$ so $Rb \neq 0$, and $Rb = R(be) \subset Re$, thus by minimality $Re = Rb$, say $e = rb$. Hence $(ere)b = er(eb) = erb = e^2 = e$, so b has a left inverse in eRe. As for the right inverse, since ere must also have a left inverse $(ere)^*$:

$$(ere)^* = (ere)^*(erb) = b$$

it follows that $b(ere) = (ere)^*(ere) = e$, and eRe is a division Ring.

□

Wedderburn Theorem

Corollary to Brauer's Lemma

Corollary.

Every nonzero left ideal in a semiprime, left artinian ring contains a nonzero idempotent.

Proof.

If $L \neq 0$ is a left ideal of R, the left artinian condition gives a minimal left ideal $K \subset L$. R is semiprime, thus $(KR)^2 \neq 0$, since KR is a non-zero ideal ($KRR \subset KR$, $RKR \subset KR$), so $0 \neq (KR)^2 = KRKR \subset KKR = K^2R$, and we have $K^2 \neq 0$. Hence Brauer's lemma applies. $\square$

$K = Re$

Wedderburn's Theorem.

If R is a simple ring with a minimal left ideal, then $R \cong M_n(D)$ for some division ring D.

Proof.

Let K be a minimal left ideal. Since R is simple, it is semiprime and by the same argument as above $K^2 \neq 0$, so by Brauer's lemma $K = Re$ where e is idempontent and $D = eRe$ is a division ring.

Then K is a right D-module and, if $r \in R$, the map $\alpha_r : K \longrightarrow K$ given by $\alpha_r(k) = rk$ is a D-linear transformation.

$$\alpha_r(k_1 + k_2) = r(k_1 + k_2) = \alpha_r(k_1) + \alpha_r(k_2)$$

$$\alpha_r(kd) = rkd = \alpha_r(k)d$$

Wedderburn Theorem

Hence $\rho : R \to \text{end}_D K$ defined by $r \mapsto \alpha_r$ is a ring homomorphism:

$$\rho(x+y)(k) = \alpha_{x+y}(k) = (x+y)(k) = \alpha_x(k) + \alpha_y(k) = (\rho(x)+\rho(y))(k)$$

$$\rho(xy)(k) = \alpha_{xy}(k) = xyk = \alpha_x(\alpha_y(k)) = (\rho(x)\circ\rho(y))(k)$$

$\rho(r) = 0$

Now we will show that this is in fact an isomporhism. Note that if $\alpha_r(k) = 0$ for all $k \in K$, then $rK = rRe = 0$, so $rR = rReR = 0$ implies $r = 0$, thus ρ is injective.

$\alpha_r(k) = 0 = rk$

$0 \neq ReR = R$

Wedderburn Theorem

To see that ρ is surjective, write $1 \in R = ReR$ as $1 = \sum_{i=1}^{n} r_i e s_i$. Given $\alpha \in \operatorname{end}_D K$, let $t = \sum_{i=1}^{n} \alpha(r_i e) e s_i$. Then the D-linearity of α gives

$K = Re$

$$\begin{aligned}
\alpha(re) &= \alpha\left(\sum_i (r_i e s_i) re\right) \\
r_i e^2 s_i re &= \sum_i \alpha([r_i e][\underbrace{e s_i re}_{eRe}]) \\
&= \sum_i \alpha(r_i e) e s_i re \\
&= tre \\
&= \alpha_t(re).
\end{aligned}$$

Since this is true for all $re \in Re$, $\alpha = \alpha_t$ and it follows that $R \cong \operatorname{end}_D K$.

Note that $e \in A$, where $\alpha_e(K) = eRe \Rightarrow$

$$A = \{x \in R \mid \dim_D \alpha_x(K) < \infty\},$$

thus A is a nonzero ideal and by simplicity $A = R$, in particular $1 \in A$ implies $\alpha_1(K) = K$ is finite dimensional, hence

$$R \cong \operatorname{end}_D K \cong M_{\dim_D K}(D).$$

□

Weddernburn-Artin Theorem

Preliminaries

Let I denote the set of idempotents in R. If $e, f \in I$, we write $e \leq f$ if $ef = e = fe$, i.e., if $eRe \subset fRf$. This is a partial ordering on I (with 0 and 1 as the least and greastest elements).

$\supset \quad \subset \qquad \subset R$

I is said to satisfy the *maximum condition* if every non-empty subset contains a maximal elements, that is, if $e_1 \leq e_2 \leq \cdots$ in I implies $e_n = e_{n+1} = \cdots$ for some $n \geq 1$. Analagously, I is said to satisfy the *minimal condition* if $e_1 \geq e_2 \geq \cdots$ in I implies $e_n = e_{n+1} = \cdots$ for some $n \geq 1$. A set of idempotents is called *orthogonal* if $ef = 0$ for all $e \neq f$ in the set.

Wedderburn-Artin Theorem

Lemma.

The following are equivalent for a ring R:

(1) R has maximum condition on idempotents.

(2) R has minimum condition on idempotents.

(3) R has maximum condition on left ideals Re, $e^2 = e$.

(4) R has minimum condition on left ideals Re, $e^2 = e$.

(5) R contains no infinite orthogonal set of idempotents.

Steps.

We will prove (1) $\iff$ (2), (3) $\iff$ (4),
(1) $\implies$ (3) $\implies$ (5) $\implies$ (1).

Wedderburn-Artin Theorem

Lemma (1) $\iff$ (2)

Proof.

Let us first note that, if $e \leq f$, we have

$ef = e = fe$

$$(1-f)(1-e) = 1 - \cancel{e} - f + \cancel{fe} = 1 - f,$$
$$(1-e)(1-f) = 1 - \cancel{e} - \cancel{f} + \cancel{ef} = 1 - f.$$

$(1-f)^2 = 1-f$

Thus $1 - f \leq 1 - e$ and the converse easily follows from the previous equalities. If $e_1 \geq e_2 \geq \cdots$ in I implies $e_n = e_{n+1} = \cdots$ for some $n \geq 1$, the above statement can be used to see that

$$1 - e_1 \leq 1 - e_2 \leq \cdots \implies 1 - e_n = 1 - e_{n+1} = \cdots.$$

This proves (1) $\iff$ (2).

$1-f \to 1-(1-f) = f$

Wedderburn-Artin Theorem

Lemma (3) $\iff$ (4)

$Re_i \subset Re_{i+1} \subset \cdots$

$e_i : R$

Similarly to the previous proof, note that

$ef = rff = rf = e$

$Re = R(ef) \subset Rf$

$$
\begin{aligned}
Re \subset Rf &\iff ef = e \\
&\iff (1-e)(1-f) = 1-f \\
&\iff (1-f)R \subset (1-e)R.
\end{aligned}
$$

Thus a ascending chain can be turned to an descending one. This proves (3) $\iff$ (4).

Wedderburn-Artin Theorem

Lemma (1) $\implies$ (3)

$Re_i \subset Re_j$

If $Re_1 \subset Re_2 \subset \cdots$ where $e_i^2 = e_i$ for each i, then $e_i e_j = e_i$ for all $j \geq i$. Inductively construct idempotents $f_1 \leq f_2 \leq \cdots$ as

$$f_1 = e_1$$

$$f_{i+1} = f_i + e_{i+1} - e_{i+1} f_i$$

$f_i \in Re_{i+1}$

Note that if $f_i \in Re_i$, then $f_{i+1} \in Re_{i+1}$, thus $f_i \in Re_i$ for all i and $f_i e_k = f_i$ for $k \geq i$. Moreover if $f_i^2 = f_i$

$$\begin{aligned} f_{i+1}^2 &= f_i^2 + f_i e_{i+1} - f_i e_{i+1} f_i + e_{i+1} f_i + e_{i+1}^2 - e_{i+1}^2 f_i \\ &\quad - e_{i+1} f_i^2 - e_{i+1} f_i e_{i+1} + e_{i+1} f_i e_{i+1} f_i \\ &= f_i + e_{i+1} - e_{i+1} f_i = f_{i+1} \end{aligned}$$

Wedderburn-Artin Theorem

lemma (1) $\implies$ (3)

Finally note that

$$f_i f_{i+1} = f_i^2 + f_i \cancel{e_{i+1}} - f_i \cancel{e_{i+1}} f_i = f_i$$
$$f_{i+1} f_i = f_i^2 + \cancel{e_{i+1} f_i} - \cancel{e_{i+1} f_i^2} = f_i$$

in other words $f_i \leq f_{i+1}$. Thus (1) implies that $f_n = f_{n+1} = \cdots$ for some n and hence that $e_{i+1} = e_{i+1} f_i \in Re_i$ for $i \geq n$. It follows that $Re_n = Re_{n+1} = \cdots$.

The maxium condition on right ideals is proved analogously by taking $f_{i+1} = f_i + e_{i+1} - f_i e_{i+1}$ instead.

Wedderburn-Artin Theorem

Lemma (3) $\implies$ (5)

$Re_1 \subset Re_2 \subset \cdots$

By contrapositive suppose we have an infinite orthogonal set of distinct idempotents $\{e_n\}$. Construct $f_n = \sum_{k=1}^{n} e_k$, then for $m < n$

$e_k e_k$

$$f_m f_n = \left(\sum_{k=1}^{m} e_k\right)\left(\sum_{k=1}^{n} e_k\right) = \sum_{k=1}^{m} e_k^2 = f_m$$

$Rf_1 \subset Rf_2 \subset \cdots$ $\qquad Rf_{n+1} = Rf_n$

Thus $f_n^2 = f_n$ and $Rf_n \subset Rf_{n+1}$. Note that if $f_{n+1} \in Rf_n$, then $f_{n+1} = rf_n$ and $f_n = f_{n+1}f_n = rf_n^2 = f_{n+1}$ implies $e_{n+1} = 0$, thus $f_{k+1} \in Rf_k$ at most once because $\{e_n\}$ are distinct, it follows that $Rf_1 \subset Rf_2 \subset \cdots$ does not terminate.

$e_1 + \cdots e_n = e_1 + \cdots + e_{n+1}$

$e_{n+1} = 0$

Wedderburn-Artin Theorem

Lemma (5) $\implies$ (1)

Suppose that $e_1 \leq e_2 \leq \cdots$ does not end. Construct $f_1 = e_1$, and $f_{n+1} = e_{n+1} - \sum_{k=1}^{n} f_k$. By induction we prove $\{f_i\}_i$ is idempotent and orthogonal. $\{f_1\}$ is idempotent and orthogonal, so suppose $\{f_i\}_{i=1}^{n}$ is an idempotent and orthogonal set, then

$e_j f_k = f_k = f_k e_j$

$j \geq k$

$$
\begin{aligned}
f_{n+1}^2 &= \left(e_{n+1} - \sum_{k=1}^{n} f_k \right)^2 \\
&= e_{n+1}^2 - e_{n+1} \sum_{k=1}^{n} f_k - \sum_{k=1}^{n} f_k e_{n+1} + \left(\sum_{k=1}^{n} f_k \right)^2 \\
&= e_{n+1} - \cancel{2} \sum_{k=1}^{n} f_k + \sum_{k=1}^{n} f_k^2 \qquad (e_{n+1} f_k = f_k) \\
&= f_{n+1}
\end{aligned}
$$

Wedderburn-Artin Theorem

Lemma (5) $\implies$ (1)

$j < n+1$

$$
\begin{aligned}
f_{n+1}f_j &= \left(e_{n+1} - \sum_{k=1}^{n} f_k\right)(f_j) \\
&= \underline{e_{n+1}f_j} - \sum_{k=1}^{n} \underline{f_k f_j} \\
&= f_j - f_j^2 \\
&= 0
\end{aligned}
$$

We have thus constructed an infinite orthogonal set of idempotents.

By contrapositive (5) $\implies$ (1). $\square$

Wedderburn-Artin Theorem

If R is a semiprime left artinian ring then.

$$R \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_r}(D_r)$$

where each D_i is a division ring and $M_n(D)$ denote the ring of $n \times n$ matrices over D.

Proof.

Let K be a minimal left ideal, let $\underline{S = KR}$ and let $Sr_1 \subset S$

$M = \{a \in R \mid Sa = 0\}$. Then S is an ideal because K is a left ideal and M is an ideal because for all $r_1, r_2 \in R$ if $a \in M$,

$\underline{S(r_1ar_2)} \subset \underline{S(ar_2)} = \underline{0}$. We then claim

$$R \cong S \times M$$

Wedderburn-Artin Theorem

First note $S \cap M = 0$ because R is semiprime and $(S \cap M)^2 \subset SM = 0$.
Define $\rho : S \times M \to R$ by $(s, m) \mapsto s + m$, this is an homomorphism:

$$
\begin{aligned}
\rho((s_1, m_1) + (s_2, m_2)) &= \rho(s_1 + s_2, m_1 + m_2) \\
&= s_1 + s_2 + m_1 + m_2 \\
&= \rho(s_1, m_1) + \rho(s_2, m_2) \\
\rho((s_1, m_1) \cdot (s_2, , m_2)) &= \rho(s_1 s_2, m_1 m_2) \\
&= s_1 s_2 + m_1 m_2 \\
&= s_1 s_2 + \underbrace{s_1 m_2 + m_1 s_2}_{\in S \cap M = 0} + m_1 m_2 \\
&= \rho(s_1, m_1)\rho(s_2, m_2).
\end{aligned}
$$

$(s_1 + m_1)(s_2 + m_2)$

Wedderburn-Artin Thorem

$s = -m$

Since $S \cap M = 0$, if $\rho(s, m) = s + m = 0$ then $s = m = 0$, hence ρ is injective.

$re \in S$

Now let $e \in S$ be a maximal idempotent (which exists by our lemma), note that $r = re + r(1 - e)$, thus for surjectivity it's enough to show $1 - e \in M$.

$Sa = 0$

If this is not the case, then $S(1 - e) \neq 0$ and by the corollary to Brauer's lemma there is a nonzero idempotent $f \in S(1 - e)$. Then $f = s(1 - e)$ means $f(1 - e) = s(1 - e)^2 = s(1 - e) = f$, thus $fe = 0$.

$f - fe = f$

Wedderburn-Artin Thorem

$fe = 0$

Let $g = e + f - ef$, we see that
$g^2 = e^2 + \cancel{ef} - \cancel{e^2f} + \cancel{fe} + f^2 - \cancel{fef} - \cancel{efe} - ef^2 + \cancel{efef} = g$ is an idempotent in S, furtheremore $e \leq g$:

$$eg = e^2 + \cancel{ef} - \cancel{e^2f} = e$$
$$ge = e^2 + \cancel{fe} - \cancel{efe} = e$$

thus by the maximality of e, we must have $\underline{e = g} = e + f - ef$, so $\underline{f = ef}$, however $f = f^2 = fef = \underline{0}$ is a contradiction. So $\underline{1 - e} \in M$ and $R \cong S \times M$.

Wedderburn-Artin Theorem

Since $1 = s_1 + m_1$, we have

$$s_1 s = \overset{1}{(s_1 + m_1)} s = s = s(s_1 + m_1) = s s_1$$
$$m_1 m = (s_1 + m_1) m = m = m(s_1 + m_1) = m m_1$$

shows that S and M are rings with unity, moreover $s_1 = e$ and $m_1 = 1 - e$, by the maximality of e in S.

If L is a left ideal of S, then $\overset{SL \subset L}{RL} \cong (S \times M) \overset{0 \times L}{(L \times 0)} \cong SL \subset L$, and the same is true for M, this means that left ideal of S and M are left ideals of R, so they inherit the hypotheses on R.

Wedderburn-Artin Theorem

$AK \subset A$ $\qquad$ $AK \subset K$

Now we'll show that S is simple. If $0 \neq A \subset S$ is an ideal, then $AK \subset A$ and $AK \subset K$ tells us that $0 \neq A^2 \subset AS = AKR \subset (A \cap K)R$, $=0$ thus $A \cap K \neq 0$ and the minimality of K gives $K \subset A$, whence $S = KR \subset AR \subset A$.

$S = A$ $\qquad$ $R \cong S \times 0$

Finally if $M = 0$ the proof is complete by Wedderburn's theorem. Otherwise we can repeat this process with R replaced by M to get $R \cong S \times \underbrace{S_1 \times M_1}_{\cong M}$, where S_1 is simple. This cannot continue indefinitely by the artinian hypothesis, so Wedderburn's theorem completes the proof. $\blacksquare$

$S \cong M_n(D)$