Lecture 14 On Schur's lemma

The famous lemma due to Schur comes from

Schur, Issai, Neue Begründung der Theorie der Gruppencharaktere, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 1905, pp 406-432.

Let K be any field and let A be a K-algebra, i.e. A is a ring with a K-vector space structure

$$\begin{aligned} K \times A &\longrightarrow A \\ (\lambda, a) &\longmapsto \lambda \cdot a \end{aligned}$$

s.t. $\forall$ $\lambda \in K$ and $a, b \in A$:

$$\lambda \cdot (ab) = (\lambda \cdot a) b = a (\lambda \cdot b).$$

Thm (Schur's lemma) Suppose S and T are simple A-modules and

$$\varphi \in \mathrm{Hom}_A(S, T).$$

Then either $\varphi = 0$ or φ is an isomorphism. In particular, if $S = T$ then

$$\mathrm{End}_A(S)$$

is a division algebra, and if we further assume that

$$k = \overline{k} \quad \text{and} \quad \dim_k S < \infty$$

then $\exists\, \lambda \in k$ s.t. $\varphi = \lambda \cdot id_S$.

Proof

Suppose $\varphi \in \mathrm{Hom}_\Lambda(S, T)$ is non-zero, so

$$0 \leq \mathrm{Ker}(\varphi) \lneq S$$

and the simplicity of S implies that $\mathrm{Ker}(\varphi) = 0$. Also

$$0 \lneq \mathrm{Im}(\varphi) \leq T$$

so the simplicity of T gives $\mathrm{Im}(\varphi) = T$. Therefore φ is an isomorphism.

Now put $S = T$ and assume further that $K = \overline{K}$ and that $\dim_K(S) < \infty$. We'll need the following

<u>Lemma</u> If V is a finite dimensional vector space over an algebraically closed field K, and $\varphi \in \mathrm{End}_K(V)$, then $\exists\ \underset{\neq 0}{v} \in V$, $\lambda \in K$ s.t.

$$\varphi(v) = \lambda v$$

Proof
[Ex.]

The lemma for $V = S$ yields

$$f := \varphi - \lambda \cdot id_S \in \mathrm{End}_A(S)$$

s.t.

$$0 \lneq \mathrm{Ker}(f) \leq S$$

$0 \neq v \in$

Thus $\mathrm{Ker}(f) = S$ by the simplicity of S, which means that $f = 0$, i.e. $\varphi = \lambda\, id_S$ $\square$

<u>Defn</u> The centre $Z(A)$ of a given K-algebra A is the subalgebra

$$Z(A) := \{ z \in A \mid \forall a \in A : za = az \}$$

<u>Remark</u> For $A = M_n(K)$ we have $Z(A) = K \cdot id$.

Lemma Suppose A an algebra over an algebraically closed field K and let S be a simple A-module. Then $\forall\ z \in Z(A)$ $\exists\ \lambda_z \in K$ s.t. $zs = \lambda_z s$, for all $s \in S$.

Proof

For each $z \in Z(A)$ have the K-linear map

$$\begin{aligned} S &\xrightarrow{\rho_z} S \\ s &\longmapsto zs \end{aligned}$$

It is also A-linear. Indeed, for each $a \in A$ we have

$$\rho_z(as) = z(as) = (za)s = (az)s = a(zs) = a\rho_z(s).$$

So by the second part of the above theorem, there is some $\lambda_z \in K$ s.t. $\rho_z = \lambda_z \, id_S$, i.e. $zs = \lambda_z s$, $\forall s \in S$ $\square$

<u>Corollary</u> If A is a commutative algebra over an a.c. field K, then every f.d. simple A-module S is 1-dimensional.

Proof

Then $A = Z(A)$, so by the lemma $\forall a \in A \; \exists \lambda_a \in K$ s.t. $as = \lambda_a s$, i.e. $As = K \cdot s$. Hence picking $s \neq 0$, $As = S$. Thus $K \cdot s = S$ $\square$

Remark The condition $K = \bar{K}$, above, is needed. Indeed, consider the $\mathbb{R}$-algebra $A := \mathbb{R}[X]/(X^2+1)\mathbb{R}[X] \cong \mathbb{C}$. Clearly $\dim_{\mathbb{R}}(A) = 2$. But $S := A$ is simple if we regard it as an A-module.

Remark The condition $\dim_K(S) < \infty$ is also needed. Indeed, just consider the $\mathbb{C}$-algebra $A = \mathbb{C}(X)$ and regard it as an A-module. It is simple but not of dimension 1 over $\mathbb{C}$.

Schur lemma and its corollary are very usefull tools.
For example, the corollary is used in proofs of the Artin-Wedderburn theorem.