Lecture 15 On the classical quaternions

The classical quaternions may be defined as the 4-dimensional $\mathbb{R}$-algebra with basis $\mathcal{B} = \{1, i, j, k\}$ and multiplication rules

$$i^2 = j^2 = k^2 = -1$$

$$ij = k$$

$$ki = j$$

$$jk = i$$

We shall denote them $\mathbb{H} := \{ h = a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \}$.

We have an isomorphism

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{H}^{op} \\ h &\longmapsto a - bi - cj - dk =: \bar{h} \end{aligned}$$

i.e. $\forall\, h_1, h_2 \in \mathbb{H}$: $\overline{h_1 h_2} = \bar{h}_2 \bar{h}_1$. Also, $\forall\, h \in \mathbb{H}$

$$h\bar{h} = a^2 + b^2 + c^2 + d^2 =: n(h)$$

Therefore $\forall$ non-zero $h \in \mathbb{H}$ we have the formula

$$h^{-1} = \frac{\bar{h}}{h\bar{h}} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}.$$

Thus $\mathbb{H}$ is a division ring.

We have a faithfull representation

$$\mathbb{H} \xrightarrow{\rho} M_2(\mathbb{C})$$

$$h = z_1 + z_2 j \longmapsto \rho_h := \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

i.e. ρ is a ring homomorphism s.t. $\operatorname{Ker}(\rho) = 0$.

Moreover, $\forall\, h \in \mathbb{H}$

$$n(h) = \det(\rho_h)$$

We have a group homomorphism

$$\mathbb{H}^{\times} \xrightarrow{n} \mathbb{R}_{>0}$$

$$h \longmapsto n(h)$$

Indeed,

$$n(h_1 h_2) = h_1 h_2 \overline{h_1 h_2} = h_1 h_2 \bar{h}_2 \bar{h}_1 = h_1 n(h_2) \bar{h}_1$$

$$= h_1 \bar{h}_1 n(h_2) = n(h_1) n(h_2).$$

As a corollary we have Euler's identity

$$(a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2) =$$

$$(a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2)^2 +$$

$$(a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2)^2 +$$

$$(a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2)^2 +$$

$$(a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)^2.$$

Let $K := \mathrm{Ker}(n) = S^3 \subseteq \mathbb{H} \cong \mathbb{R}^4$

$$h \longmapsto (a, b, c, d)$$

and

$$V := \langle i, j, k \rangle_{\mathbb{R}} \cong \mathbb{R}^3$$

$$bi + cj + dk \mapsto (b, c, d)$$

We have a group homomorphism

$$K \xrightarrow{\sigma} \mathrm{Aut}_{\mathbb{R}}(V) \cong GL_3(\mathbb{R})$$

$$h \longmapsto \left(\begin{array}{c} V \xrightarrow{\sigma_h} V \\ v \longmapsto h v h^{-1} \end{array} \right)$$

Indeed, $\sigma_{h_1 h_2}(v) = h_1 h_2 v (h_1 h_2)^{-1}$

$$= h_1 h_2 v h_2^{-1} h_1^{-1}$$

$$= \sigma_{h_1}(\sigma_{h_2}(v))$$

$$= (\sigma_{h_1} \circ \sigma_{h_2})(v),$$

$\forall v \in V$. Therefore

$$\sigma_{h_1 h_2} = \sigma_{h_1} \circ \sigma_{h_2}.$$

Moreover, $\sigma(K) \subseteq SO(3)$. Indeed, $\forall h \in K$ and $v \in V$ we have

$$
\begin{aligned}
n(\rho_h(v)) &= n(hvh^{-1}) \\
&= n(h)\,n(v)\,n(h^{-1}) \\
&= n(h)\,n(v)\,n(h)^{-1} \\
&= n(v)
\end{aligned}
$$

It remains to show that $\det(\rho_h) = 1$. [Ex.]

It is not difficult to show that the group homomorphism

$$K \xrightarrow{\sigma} SO(3)$$

is $2-1$. In the literature K is known as $Spin(3)$ as it is the universal cover of $SO(3)$, where

$$SO(3) \overset{top}{\cong} \mathbb{P}^3(\mathbb{R}),$$

and $\pi_1(\mathbb{P}^3(\mathbb{R})) \cong \{-1, 1\}$.

It turns out that Spin (4) $= K \times K$. Indeed, by the work of Cayley (1854) we have

$$\begin{aligned} \mathrm{Spin}(4) &\xrightarrow{\ \rho\ } SO(4) \\ (A_+, A_-) &\longmapsto \left(\begin{array}{c} \mathbb{H} \xrightarrow{\ \rho_{A_+,A_-}\ } \mathbb{H} \\ h \longmapsto A_+ h A_-^{-1} \end{array} \right) \end{aligned}$$

This is a 2 - 1 group homomorphism. [Ex.]

Each $L \in SO(3)$ there are planes $P_1, P_2 \subseteq \mathbb{H}$ s.t

(i) $P_1 \perp P_2$ and $P_1 \oplus P_2 = \mathbb{H}$

(ii) $L P_i = P_i$, so $L_i := L | P_i \in SO(2)$, $i = 1, 2$.

Let θ_i be the angle determined by L_i, $i = 1, 2$.

We say that L is isoclinic if $\theta_1 = \theta_2$.

<u>Propn</u> The set of isoclinic L form a subgroup $\cong Spin(3)$.