Lecture 16 Quaternion algebras

Let V be a vector space over a field k. A quadratic space is a pair (V, q), where $q : V \longrightarrow k$ is a function s.t.

(i) $\forall\, v \in V,\ a \in k:\quad q(av) = a^2 q(v)$

(ii) the function
$$\begin{aligned} V \times V &\longrightarrow k \\ (v, w) &\longmapsto q(v+w) - q(v) - q(w) \end{aligned}$$

bilinear over k. From now on we shall assume $\chi(k) \neq 2$ so that the scalar product $v \cdot w := \frac{1}{2}\left(q(v+w) - q(v) - q(w) \right)$ attached to q; it is s.t. $v \cdot v = q(v)$, $\forall\, v \in V$.

The matrix of q with respect a basis $\mathcal{B} = \{\beta_1, \dots, \beta_n\}$ of V is

$$[q]_{\mathcal{B}} := \begin{pmatrix} \beta_1 \cdot \beta_1 & \cdots & \beta_1 \cdot \beta_n \\ \vdots & & \\ \beta_n \cdot \beta_1 & & \beta_n \cdot \beta_n \end{pmatrix}$$

and it is s.t.

$$q(v) = \sum_{i,j=1}^{n} (\beta_i \cdot \beta_j) v_i v_j ,$$

where $v = v_1 \beta_1 + \dots + v_n \beta_n$ over k. If $\mathcal{B}' = \mathcal{B} M$, $M \in GL_n(k)$, then $[q]_{\mathcal{B}'} = M [q]_{\mathcal{B}} M^t$, so

$$\det([q]_{\mathcal{B}'}) = \det([q]_{\mathcal{B}}) \det(M)^2.$$

Defn If $v, w \in V$ are such that $v \cdot w = 0$ then we say that v and w are orthogonal. We express this relationship $v \perp w$. We also write

$$v^{\perp} = \{ w \in V : v \cdot w = 0 \}$$

If the orthogonal complement $V^{\perp} = 0$ then we say that q is nondegenerate / regular. We may see that

$$q \text{ n.d.} \iff \det([q]_{\mathcal{B}}) \neq 0$$

In this case this is an element in $k^{\times} / (k^{\times})^2$ which depends only on q we denote $d(V)$. We write $d(V) = 0$, otherwise.

Def'n A vector $v \in V - \{0\}$ is isotropic if $q(v) = 0$ and call V isotropic. If V is not isotropic, then we say V is anisotropic.

Def'n A quadratic space (V, q) s.t. $\exists$ basis $\mathcal{B} = \{\beta_1, \beta_2\}$ for V where β_1 and β_2 are isotropic and s.t. $\beta_1 \beta_2 \neq 0$ is called a hiperbolic plane. After multiplying β_2 by $\frac{1}{\beta_1 \cdot \beta_2}$ we may assume that $\beta_1 \cdot \beta_2 = 1$, so that

$$[q]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In particular, all hyperbolic planes isometric, regular, with $\det(q) = -1$, and also $\forall\ a_1, a_2 \in k$

$$q(a_1\beta_1 + a_2\beta_2) = 2a_1a_2(\beta_1 \cdot \beta_2),$$

hyperbolic plane is *universal* i.e. $q(V) = k$.

<u>Prop'n 1</u> If $\dim(V) = 2$ then the following are eq.

(1) V is a hyperbolic plane

(2) V is isotropic & regular

(3) $d(V) = -1$

Proof

$(1) \Rightarrow (2)$: follows from $[q]_{\mathcal{B}}$ with $\mathcal{B}$ as above.

$(2) \Rightarrow (3)$: Pick $v_1 \in V$ isotropic and let $v_2 \in V$ s.t.

$\mathcal{B}' := \{v_1, v_2\}$ is a basis of V and

$$[q]_{\mathcal{B}'} = \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M_2(k),$$

so $d(V) = -a^2 \underset{(\text{reg})}{\neq} 0$, so $d(V) = -1$.

(3) $\Rightarrow$ (1): In particular we have V regular and thus

$q(V) \neq 0$, so $\exists$ non zero $a \in q(V)$ and $v_1 \in V$ s.t.

$$q(v_1) = a.$$

Hence by the regularity of V there $\exists$ $v_2 \in V$ s.t.

$$V = k v_1 \hat{\oplus} k v_2.$$

But $d(V) = -1$, so $-q(v_1)q(v_2) \in (k^\times)^2$ and thus

$\mathcal{B} = \left\{ \frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2} \right\}$ is a basis of V s.t.

$$[q]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \square$$

Prop'n 2 If V is a regular isotropic space then

$$V = \mathcal{U} \hat{\oplus} V'$$

where $\mathcal{U}$ is a hyperbolic plane and hence V is universal.

Proof

Let $v_1 \in V$ isotropic. By the regularity of V, $\exists\, v_2 \in V$ s.t. $v_1 \cdot v_2 \neq 0$ and $\mathcal{U} := k v_1 \oplus k v_2$ is a regular isotropic space. So by Prop'n 1 it is a hyperbolic plane $\square$

Propn 3 Let V be regular and $a \in k$. Then

$$a \in q(V) \iff \langle -a \rangle \hat{\oplus} V \text{ is isotropic.}$$

Proof

[We'll get back to this soon.]

Prop'n 4 If $U \subseteq V$ is regular s.t. $\dim(U) = 3$ and $\dim(V) = 4$ with $d(V) = 1$, then V isotropic $\Longleftrightarrow$ U isotropic.

Proof

($\Longleftarrow$): Clear.

($\Longrightarrow$): Write $V = U \mathbin{\hat{\oplus}} \langle a \rangle$, $a \in k^\times$, $\overset{\text{(Prop'n 3)}}{\Longrightarrow}$ $-a \in q(U)$.

Thus $U = P \mathbin{\hat{\oplus}} \langle -a \rangle$, $\dim(P) = 2$. So

$$V = P \mathbin{\hat{\oplus}} \langle -a \rangle \mathbin{\hat{\oplus}} \langle a \rangle,$$

hence $1 = d(V) = -d(P)$, so $dP = -1$ and P is an hyperbolic plane by Prop'n 1. Hence U is isotropic $\square$

We'll define $\forall\ a,b \in k^\times$ the quaternion algebra $\left(\frac{a,b}{k}\right)$ as the 4-dim'l k-algebra with basis $\mathcal{B} = \{1, i, j, k\}$ s.t.

$\times$	t_2	$x_2 i$	$y_2 j$	$z_2 k$
t_1	$t_1 t_2$	$t_1 x_2 i$	$t_1 y_2 j$	$t_1 z_2 k$
$x_1 i$	$x_1 t_2 i$	$x_1 x_2 a$	$x_1 y_2 k$	$x_1 z_2 aj$
$y_1 j$	$y_1 t_2 j$	$-y_1 x_2 k$	$y_1 y_2 b$	$-y_1 z_2 bi$
$z_1 k$	$z_1 t_2 k$	$-z_1 x_2 aj$	$z_1 y_2 bi$	$-z_1 z_2 ab$

We have

$$\alpha = t + xi + yj + zk ,$$

$$\bar{\alpha} = t - xi - yj - zk$$

$$n(\alpha) = \alpha\bar{\alpha} = t^2 - ax^2 - by^2 + abz^2$$

$$t(\alpha) = \alpha + \bar{\alpha} = 2t$$

where $t, x, y, z \in k$.

The k-algebra $\left(\frac{a,b}{k}\right)$ may be regarded as a subalgebra of $M_2(k)$ by making the identifications

$$i := \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad j := \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \in M_2(K),$$

where $\alpha \in K$ is a root of $f(X) = X^2 - a$ and $\beta \in K$ is a root of $g(X) = X^2 + b$ and K/k is a field extn where both f and g split. Clearly $i^2 = a I_2$, $j^2 = b I_2$ and $ij = \begin{pmatrix} 0 & \alpha\beta \\ \alpha\beta & 0 \end{pmatrix} = -ji$.

Equip the k-algebra $\left(\frac{a,b}{k}\right)$ with the k-bilinear map

$$\langle \alpha, \beta \rangle = t(\alpha\bar{\beta}) \in k$$

$\forall\ \alpha, \beta \in \left(\frac{a\,b}{k}\right)$.

Ex Show that $\langle \cdot , \cdot \rangle$ is two-times the symmetric bilinear map attached to $n(\cdot)$.

Write $B := \left(\frac{a,b}{k}\right)$ and $B_0 = \{x \in B : t(x) = 0\}$.

Ex Show that $B = k \oplus B_0$ as k-vector space.

Thm The following are equivalent

(a) The algebra $B = \left(\frac{a,b}{k}\right)$ is not a division algebra.

(b) B is isotropic.

(c) B_0 is isotropic.

(d) $(a,b)_k = 1$.

Here for $a, b \in k^\times$ we let

$$(a,b)_k := \begin{cases} 1, & \text{if } ax^2+by^2=z^2 \text{ has a non-trivial sol'n } (x,y,z) \in k \times k \times k. \\ -1, & \text{otherwise.} \end{cases}$$

This is known as the *Hilbert symbol* and it was introduced in the stament of the Hilbert reciprocity formula.

Proof

$(b) \Rightarrow (c)$; $\exists\, x \in B - \{0\}$ s.t. $n(x) = 0$; WLOG $t(x) \neq 0$. Then $\exists\, y \in B_0 \cap x^{\perp}$ s.t. $y \neq 0$. Consider that

$$0 = \langle x, y \rangle = t(x\bar{y}) = -t(xy).$$

Therefore

$$xy \in B_0. \qquad q(\alpha) = \langle \alpha, \alpha \rangle = 2\, n(\alpha)$$

<u>Case 1</u> $xy = 0$: Recall $t(x) = x + \bar{x}$, so

$$\bar{x}y = t(x)y - xy.$$

Hence $B_0 \ni \bar{x}y = t(x)y \neq 0$. But $0 = n(x) = n(\bar{x})$,

so $n(\bar{x}y) = n(\bar{x})n(y) = 0$.

Case 2 $xy \neq 0$: We have shown that $xy \in B_0$.

Also
$$n(xy) = n(x)n(y) = 0.$$

In both cases we have produced a non-zero element $z \in B_0$

s.t. $n(z) = 0$ and (b) $\Rightarrow$ (c) follows.

(c) $\Rightarrow$ (b) : Trivial.

$\neg$(a) $\Leftrightarrow \neg$(b) : Let $x \in B$ and suppose x^{-1} exists.

Then $x^{-1} n(x) = x^{-1} x \bar{x} = \bar{x} \neq 0$, so $n(x) \neq 0$.

Conversely, if $n(x) \neq 0$, then $n(x)$ is invertible in k, so we may write $\bar{x}\, n(x)^{-1}$, which is obviously the inverse of x.