Lecture 17 Quivers and Categories

The group algebra construction may be understood quite profitably under the light of category theory, which is basically quiver theory with extra structure. So first we'll recall some definitions of the latter theory.

A *quiver*[1] is a <u>directed</u> graph where loops and multiple arrows are allowed, as in the *Kronecker quiver*

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$$

This theory was developed in the context of representation theory in Peter Gabriel, *Unzerlegbare Darstellungen I*, Manuscripta Mathematica 6 pp 71–103 (1972).

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1 *Köcher* in German.

We may define a quiver as an ordered quadruple

$$\Gamma = (V, E, s, t)$$

where we call V the set of vertices, E the set of edges, and

$$E \xrightarrow{s} V,$$

$$E \xrightarrow{t} V$$

are the functions giving the start and the end (i.e. the orientation) each edge.

A morphism $\Gamma \xrightarrow{f} \Gamma'$ from a quiver Γ to a quiver Γ' is defined by an ordered pair $f = (f_v, f_e)$ when $V \xrightarrow{f_v} V'$ and $E \xrightarrow{f_e} E'$ are funtions that make the following diagrams

$$\begin{tikzcd} V \arrow[r, "f_v"] & V' \\ E \arrow[u, "s"] \arrow[r, "f_e"] & E' \arrow[u, "s'"'] \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "f_v"] & V' \\ E \arrow[u, "t"] \arrow[r, "f_e"] & E' \arrow[u, "t'"'] \end{tikzcd}$$

commute. $f_v \circ s = s' \circ f_e$, $f_v \circ t = t' \circ f_e$

A small category is a quiver $\mathcal{C}$ where we denote $ob(\mathcal{C})$ the set of vertices and $\forall\ A, B \in ob(\mathcal{C})$ we write

$$\mathrm{Hom}(A, B) := \{ \text{arrows from } A \text{ to } B \}$$

and $\forall\ A, B, C \in ob(\mathcal{C})$ we have a map

$$\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(A,C)$$

$$\left(A \xrightarrow{f} B,\ B \xrightarrow{g} C\right) \longmapsto \begin{tikzcd} A \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

s.t.

(1) $\forall\, A, B, C, D \in ob(\mathcal{C})$, $\forall\, f \in Hom(A,B)$, $g \in Hom(B,C)$, $h \in Hom(C,D)$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

(2) $\forall\, A \in ob(\mathcal{C})\ \exists\, e_A \in Hom(A,A)\ \forall\, B \in ob(\mathcal{C})$

(2.1) $e_A \circ f = f$, $\forall\, f \in Hom(B,A)$

(2.2.) $f \circ e_A = f$, $\forall\, f \in Hom(A,B)$

Such e_A is clearly unique in $Hom(A,A)$, $\forall\, A \in ob(\mathcal{C})$.
It is denoted 1_A.

Given small categories $\mathcal{C}$ and $\mathcal{C}'$ a morphism

$$\mathcal{C} \xrightarrow{F} \mathcal{C}'$$

is a quiver morphism s.t. $\forall\, A, B, C \in ob(\mathcal{C})$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \arrow[dl, phantom, "\circlearrowleft"] \\ & C \end{tikzcd} \qquad F(g\circ f) = F(g)\circ F(f) \qquad \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[dr] & F(B) \arrow[d, "F(g)"] \arrow[dl, phantom, "\circlearrowleft"] \\ & F(C) \end{tikzcd}$$

and s.t. $\forall\, A \in ob(\mathcal{C}) : F(1_A) = 1_{F(A)}$.

A morphism of categories $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ is also known as a *functor*.

If we allow $\mathrm{ob}(\mathcal{C})$ to be a class, e.g. the class of all sets, then we drop the term 'small' and call $\mathcal{C}$ a category.

Given categories $\mathcal{C}$ and $\mathcal{C}'$, and functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathcal{C}' \end{tikzcd}$$

a natural transformation τ from F to G is as follows. For all arrow f of $\mathcal{C}$

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowright"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\tau_B"] & G(B) \end{tikzcd}$$

We say that the categories $\mathcal{E}$ and $\mathcal{E}$ are equivalent if there are functors $\mathcal{E} \xrightarrow{F} \mathcal{E}'$ and $\mathcal{E}' \xrightarrow{G} \mathcal{E}$ s.t.

(i) $G \circ F \overset{\tau}{\xrightarrow{\sim}} 1_{\mathcal{E}}$

(ii) $F \circ G \overset{\tau'}{\xrightarrow{\sim}} 1_{\mathcal{E}'}$

for suitable natural transformations τ and τ', where $1_{\mathcal{D}}$ denotes the identity functor of any category $\mathcal{D}$.