Lecture 18 The group algebra $k[G]$

<u>Def'n</u> Given any field k and any set S, the k-vector space $\langle S \rangle_k$ generated by S is

$$\langle S \rangle_k := \{ \text{finitely supported functions } S \longrightarrow k \}.$$

The canonical basis $\mathcal{B}$ of $\langle S \rangle_k$ is

$$\mathcal{B} = \{ e_P \mid P \in S \},$$

where $e_P(Q) = \delta_{P,Q}$. Here as usual,

$$\delta_{P,Q} := \begin{cases} 1, & \text{if } P = Q \\ 0, & \text{if } P \neq Q \end{cases} \qquad (\text{Kronecker's } \delta)$$

Therefore $\forall f \in \langle S \rangle_k$ we have a unique expression

$$f = \sum_{P \in S} a_P e_P, \qquad a_P \in k.$$

<u>Defn</u> Given any group G and any field k, the group algebra[1] $k[G]$ is the k-vector space $\langle G \rangle_k$ generated by G, equipped with the multiplication rule

$$\left(\sum_{g\in G} a_g e_g\right)\left(\sum_{g\in G} b_g e_g\right) := \sum_{g\in G} c_g e_g ,$$

where $c_g := \sum_{hk=g} a_h b_k$.

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1 Schur, I., *Neue Begründung der Theorie der Gruppencharaktere*, pp 406-432, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 1905.

The structure $k[G]$ thus defined is that of a k-algebra. Moreover, for each group G, we have thus defined a category of $k[G]$-modules, whose objects are $k[G]$-modules V and whose morphisms are the k-linear maps

$$V \xrightarrow{\varphi} W$$

that are also $k[G]$-linear. Let's denote this category

$$k[G]-\mathrm{Mod}.$$

It turns out that we have an equivalence of categories

$$\mathrm{Rep}_k(G) \xrightarrow{\sim} k[G]\text{-Mod}$$

where $\mathrm{Rep}_k(G)$ is the category of k-linear representations of G. Indeed, to each k-linear rep'n

$$\begin{aligned} \rho : G &\longrightarrow GL(V) \\ g &\longmapsto \rho_g \end{aligned}$$

we attach the $k[G]$-module structure for V defined by letting $\forall\, \alpha \in k[G]$, $v \in V$ the action

$$\alpha \cdot v := \sum_{g \in G} a_g \, e_{g.v},$$

where $\alpha = \sum_{g \in G} a_g e_g$ and $g.v := \rho_g(v)$. Conversely, given a $k[G]$-module V, restricting the action to G yields a a k-linear representation of G.

We may see that, in particular, we thus have a 1-1 correspondence

$$\left\{ \begin{array}{l} \text{irred'l rep's of } G \\ \text{on a f.d } k\text{-vector} \\ \text{space } V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{simple } k[G]\text{-modules } V, \\ \text{where } \dim_k(V) < \infty \end{array} \right\}$$

This correspondence will link the Artin-Wedderburn thm with the rep's theory the finite groups G.