Lecture 19 On Maschke's theorem

Let G be a finite group. We'll discuss linear representations of G on a finite dimensional k-vector space V, that is, g.h.

$$G \xrightarrow{\rho} \mathrm{Aut}_k(V) \overset{\mathcal{B}}{\cong} GL_n(k)$$

Thm (Maschke, 1898) If $\chi(k) \nmid |G|$ then for each subrepresentation $U \subseteq V$ there is a subrepresentation $W \subseteq V$ s.t.

$$V = W \oplus U.$$

Proof

From the linear algebra course, $\exists$ a k-linear subspace $W' \leq V$ s.t. $V = \mathcal{U} \oplus W'$. Define the projection π' along W' by

$$\begin{tikzcd}
V \arrow[dd, "\pi'"'] & = \mathcal{U} \oplus W' & \mathcal{U} \times W' \arrow[l, "\sim"'] \\
& v := u + w' \arrow[d, mapsto] & (u, w') \arrow[l, mapsto] \\
\mathcal{U} & u &
\end{tikzcd}$$

Now we may define $\pi(v) := \frac{1}{|G|} \sum_{g \in G} (g^{-1} \pi' g) \cdot v$, $\forall v \in V$.

We'll show that $W := \ker(\pi)$ does the trick. Indeed, We have $\pi|_{\mathcal{U}} = id_{\mathcal{U}}$, so $W \cap \mathcal{U} = \{0\}$ and thus the map π is onto and $V = \mathcal{U} \oplus W$ over k.

Ex. Prove that indeed $\pi|_{\mathcal{U}} = id_{\mathcal{U}}$.

Now we need to show that W is a subrepresentation of V.
But $\forall\, h \in G,\ v \in V$

$$\pi(hv) = \frac{1}{|G|}\sum_{g\in G} g^{-1}\pi(ghv) \overset{(k := gh)}{=} \frac{1}{|G|}\sum_{k\in G}\left(hk^{-1}\pi k\right)v = h\,\pi(v).$$

I.e. π is a G-map and thus W is a subrep'n of V $\square$

Since we assumed that V is a finite dimensional k-vector space, then we have the following.

<u>Corollary</u> Notation as above, we have

$$V = \bigoplus_{i=1}^{n} V_i ,$$

where $V_1, \dots, V_n$ are irreducible subrepresentations of V.

In other words, ρ is semisimple.

Assume that $k = \mathbb{C}$.

<u>Thm</u> Let G be a finite group. Then

$$|G| = \sum_{i=1}^{g} \dim_{\mathbb{C}}(V_i)^2,$$

where $V_1, \ldots, V_g$ are the finite dimensional complex irred'l representations (up to isomorphism) of G.

Proof

Consider $V = \mathbb{C}[G]$, regarded as a $\mathbb{C}[G]$-module.

By Maschke's th'm we have a decomposition

$$V = V_1 \oplus \dots \oplus V_g$$

where each V_i is irreducible. But irreducible spaces V_i are in fact simple $\mathbb{C}[G]$-modules, so Wedderburn's th'm says that

$$V_i \cong M_{n_i}(D_i),$$

for $n \in \{1, 2, \dots\}$ and a division algebra $D/\mathbb{C}$. We claim that each $D_i \cong \mathbb{C}$. Indeed, consider the following.

Lemma Let D be a k-algebra. We have the implication

$$\left.\begin{array}{l}\text{(i) } D \text{ is a division ring.}\\ \text{(ii) } D \text{ is finite dimensional over } k\\ \text{(iii) } k \text{ is algebraically closed}\end{array}\right\} \Rightarrow D = k$$

Proof

$(\supseteq)$: ✓

$= \langle 1, x, x^2, \dots \rangle_k$

$(\subseteq)$: Let $x \in D$. Clearly (i) implies that $A := k[x]$ is an integral domain, which together with (ii) says that $A = k(x)$ is an algebraic field extension of k. Thus by (iii) we have $A = k$. So in particular, $x \in k$ $\square$

The fundamental theorem of algebra says that $\mathbb{C}$ is algebraically closed. Therefore the claim follows and we thus get

$$V_i \cong M_{n_i}(\mathbb{C}).$$

But $\dim_{\mathbb{C}}(V_i) = n_i^2$, and also $\dim_{\mathbb{C}}(\mathbb{C}[G]) = |G|$.

The theorem follows $\square$