Lecture 20 Key irred'l linear rep'ns of S_n.

Let $\mathcal{B} := \{ e_1, \ldots, e_n \}$ be the canonical basis of $V = \mathbb{C}^n$, where $n \in \{1, 2, 3, \ldots\}$. Note that for each $\pi \in S_n$ the bijective map

$$\begin{aligned} \mathcal{B} &\longrightarrow \mathcal{B} \\ e_i &\longmapsto e_{\pi(i)} \end{aligned}$$

extends to an invertible $\mathbb{C}$-linear map $V \xrightarrow{\rho_\pi} V$. Moreover, we have group homomorphism

$$\begin{aligned} S_n &\xrightarrow{\rho} \mathrm{Aut}_{\mathbb{C}}(V). \\ \pi &\longmapsto \rho_\pi \end{aligned}$$

Thus we have a rep'n of $G := S_n$ on V and consequently a G-set structure for V,

$$G \times V \longrightarrow V$$
$$(\pi, v) \longmapsto \pi \cdot v := \rho_\pi(v)$$

Recall that a $\mathbb{C}$-linear subspace $W \subseteq V$ is a *subrepresentation* of V if $G \cdot W \subseteq W$. Recall also that V is *irreducible* if the subrep'ns W of V are just the trivial ones i.e. either $W = 0$ or $W = V$.

Prop'n The above rep'n decomposes as a direct sum of subrepresentations

$$V = V_1 \oplus V_2 , \qquad \bigstar$$

where $\dim_{\mathbb{C}}(V_1) = n-1$ and $\dim_{\mathbb{C}}(V_2) = 1$. In fact $V_1 = \langle \mathcal{B}' \rangle_{\mathbb{C}}$, where $\mathcal{B}'$ is the linearly independent set

$$\mathcal{B}' := \{ e_1 - e_2 , e_2 - e_3 , e_3 - e_4 , \dots , e_{n-1} - e_n \}.$$

Moreover, V_1 and V_2 are irreducible.

The $(n-1)$-dimensional rep'n above is known as the standard rep'n of S_n

Proof

Put $V_2 := \langle \overbrace{e_1 + e_2 + \dots + e_n}^{v_0} \rangle_{\mathbb{C}}$ and note that $\forall\, \pi \in S_n$ we have

$$\pi \cdot v_0 = e_{\pi(1)} + \dots + e_{\pi(n)} = v_0 .$$

So the action of S_n on V_2 is trivial. So of course it is a subrep'n of V. Now consider

$$V_1 := \{ v \in V \mid v \cdot v_0 = 0 \}$$

We claim that V_1 is a subrep'n of V. Indeed, if $v = (x_1, \ldots, x_n) \in V_1$ and $\pi \in S_n$ then

$$\pi \cdot v = (x_{\pi(1)}, \ldots, x_{\pi(n)}),$$

but $x_{\pi(1)} + \cdots + x_{\pi(n)} = x_1 + \cdots + x_n = 0.$

Thus $\pi \cdot v \in V_1$ and our claim follows.

From basic linear algebra we see that (A) is a direct sum decomposition. It is also plain that V_2 is irreducible. The irreducibility of V_1 shall be addressed next week. $\square$

Another key irred'l rep'n of S_n is constructed as follows. Recall that we have a group homomorphism

$$\begin{aligned} S_n &\longrightarrow \{-1, 1\} \\ \pi &\longmapsto s(\pi) \end{aligned}$$

Here $s(\pi)$ denotes the sign of the permutation π,

$$s(\pi) = (-1)^{\ell(\pi)},$$

where $\pi = \tau_1 \circ \cdots \circ \tau_{\ell(\pi)}$, for transpositions $\tau_i \in S_n$.

NB. If $\ell(\pi)$ and $\ell'(\pi)$ are lengths as above, then $\ell(\pi) \equiv \ell'(\pi) \pmod 2$, so $s(\pi)$ is well-defined.

Now we may introduce the sign rep'n of S_n.

$$\begin{aligned} S_n &\longrightarrow \operatorname{Aut}_{\mathbb{C}}(V), \\ \pi &\longmapsto \left(\begin{array}{c} V \longrightarrow V \\ v \longmapsto s(\pi)\, v \end{array} \right) \end{aligned}$$

where $V = \mathbb{C}$.