Lecture 21 Key constructions

Given a set S and a ring R, the free R-module generated by S is

$$\langle S \rangle_R := \{ \text{finitely supported functions } S \longrightarrow R \}.$$

We put $\mathcal{B} := \{ e_P \mid P \in S \}$, where $e_P(Q) = \delta_{P,Q}$.

Here as usual,

$$\delta_{P,Q} := \begin{cases} 1, & \text{if } P = Q \\ 0, & \text{if } P \neq Q \end{cases}$$

Therefore $\forall\, x \in \langle S \rangle_R$ we have a unique expression

$$x = \sum_{p \in S} a_p e_p \,, \qquad a_p \in R \quad \text{almost all } 0$$

Moreover, $\forall$ R-module M and map $S \xrightarrow{f} M$

$\exists!$ R-linear map $\hat{f} : \langle S \rangle_R \longrightarrow M$ s.t. the diagram

$$\begin{tikzcd}
x \arrow[rrrr, mapsto, bend left] & \langle S \rangle_R \arrow[rr, "\hat{f}"] & & M & \sum_{p \in S} a_p f(e_p) \\
e_p & & S \arrow[ul, "\iota"] \arrow[ur, "f"'] & & \\
& p \arrow[ul, mapsto] & & &
\end{tikzcd}$$

commutes, i.e. $f = \hat{f} \circ \iota$. $\quad \bigstar$

Given R-modules M and N, their tensor product over R is

$$M \otimes_R N := \langle M \times N \rangle_R \, / \, B$$

where is the B is the R-submodule of $\langle M \times N \rangle_R$ generated by the elements of the form

$$(a_1 x_1 + a_2 x_2 , y) - a_1 (x_1, y) - a_2 (x_2, y),$$

$$(x, a_1 y_1 + a_2 y_2) - a_1 (x, y_1) - a_2 (x, y_2),$$

$\forall \; x, x_1, x_2 \in M \, ; \; y, y_1, y_2 \in N, \; a_1, a_2 \in R.$

We thus have the canonical map

$$\begin{aligned} M \times N &\xrightarrow{\kappa} M \otimes_R N \\ (x, y) &\longmapsto \kappa(x, y) =: x \otimes y \end{aligned}$$

s.t. $\forall\ x, x_1, x_2 \in M\ ;\ y, y_1, y_2 \in N,\ a_1, a_2 \in R$:

$$(a_1 x_1 + a_2 x_2) \otimes y = a_1 (x_1 \otimes y) + a_2 (x_1 \otimes y)$$

$$x \otimes (a_1 y_1 + a_2 y_2) = a_1 (x \otimes y_1) + a_2 (x \otimes y_2)$$

Therefore every element of $M \otimes_R N$ is of the form

$$\sum_{ij} x_i \otimes y_j ,$$

where $\{x_i\}_{i \in I} \subseteq M$ and $\{y_j\}_{j \in J} \subseteq N$ almost all 0.

Recall that given R-modules M, N, L, a map

$$f : M \times N \longrightarrow L$$

is R-bilinear if $\forall \; x, x_1, x_2 \in M; \; y, y_1, y_2 \in N;$
$a_1, a_2 \in R$

$$f(a_1 x_1 + a_2 x_2, y) = a_1 f(x_1, y) + a_2 f(x_1, y)$$

$$f(x, a_1 y_1 + a_2 y_2) = a_1 f(x, y_1) + a_2 f(x, y_2)$$

We may see that $\forall$ bilinear map $f : M \times N \longrightarrow L$

$\exists!$ linear map $\hat{f} : M \otimes_R N \longrightarrow L$ s.t. the diagram

$$\begin{tikzcd}
M \otimes_R N \arrow[rr, "\hat{f}"] & & L \\
M \times N \arrow[u, "\kappa"] \arrow[urr, "f"'] & &
\end{tikzcd}$$

commutes, i.e. $f = \hat{f} \circ \kappa$

A'

Remark Given a category $\mathcal{C}$, a universally repelling object $R \in ob(\mathcal{C})$ s.t. $\forall A \in ob(\mathcal{C})$

$$|\mathrm{Hom}_{\mathcal{C}}(R, A)| = 1.$$

A universally repelling object R, if it exists, is determined up to a unique isomorphism. The properties $(\star)$ and $(\star')$ may be stated in these terms, for suitably defined categories.

Given linear maps $f_i : M_i \to N_i$, $i = 1, 2$, we may define their tensor product

$$M_1 \otimes M_2 \xrightarrow{f_1 \otimes f_2} N_1 \otimes N_2$$

$$\sum_{ij} x_i \otimes y_j \longmapsto \sum_{ij} f_1(x_i) \otimes f_2(y_j)$$

If $M_i = N_i \cong k^{n_i}$ then $f_1 \otimes f_2$ is the Kronecker product of the corresponding matrices

E.g. if

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = [f_1]_{\mathcal{B}},$$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = [f_2]_{\mathcal{B}},$$

Then

$$[f_1 \otimes f_2]_{\mathcal{B} \otimes \mathcal{B}} = \begin{bmatrix} a_{11}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ & \\ a_{21}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix}.$$

Consider a k-vector space V and pick a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ for V. The map $e_i \otimes e_j \longmapsto e_j \otimes e_i$ extends to an involution ϑ of $V \otimes_k V$, which induces a direct sum decomposition

$$V \otimes_k V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V),$$

where $\mathrm{Sym}^2(V)$ is the $+1$-eigenspace and $\mathrm{Alt}^2(V)$ is the -1-eigenspace of ϑ. We have

$$\dim_k \mathrm{Sym}^2(V) = \frac{n(n+1)}{2}, \quad \dim_k \mathrm{Alt}^2(V) = \frac{n(n-1)}{2}$$

and their sum is $\dim_k (V \otimes V) = n^2$.

If $R = \mathbb{C}$, a *sesquilinear form* is a map

$$V \times V \xrightarrow{f} \mathbb{C}$$

$$(x, y) \mapsto f(x, y) =: (x \mid y)$$

s.t. $\forall\ x, x_1, x_2, y, y_1, y_2 \in V,\ a_1, a_2 \in \mathbb{C}$

$$f(a_1 x_1 + a_2 x_2, y) = a_1 f(x_1, y) + a_2 f(x_2, y)$$

$$f(x, a_1 y_1 + a_2 y) = \bar{a}_1 f(x, y_1) + \bar{a}_2 f(x, y_2)$$

We say that V is an *inner product space* if we further assume that $(x \mid x) \in \mathbb{R}_{>0},\ \forall\ x \in V - \{0\}$.

Let G be a finite group and ρ a linear rep'n of G in a finite dimensional vector space V over a field k. In other words, we have a group homomorphism

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{Aut}_k(V) \\ g &\longmapsto \rho_g \end{aligned},$$

which determines an action of G on V

$$\begin{aligned} G \times V &\longrightarrow V \\ (g, v) &\longmapsto g \cdot v := \rho_g(v) \end{aligned}.$$

From now on we shall assume that $k = \mathbb{C}$.

Recall that Maschke's thm says that for each subrepresentation $W \leq V$ there is another subrepresentation $W' \leq V$ s.t. $V = W \oplus W'$.

We may furnish a new way to get W' by endowing V with an inner product space structure s.t. $\forall\, g \in G;\ x, y \in V$:

$$(g \cdot x \mid g \cdot y) = (x \mid y).$$

Indeed, such inner product structure may be obtained by first identifying $V \overset{\mathcal{B}}{\cong} \mathbb{C}^n$ and replacing the inner product $(x|y)^{old}$ coming from this identification by

$$(x|y) := \sum_{g \in G} (g.x \mid g.y)^{old},$$

and then define $W' := W^0$ with respect to $(\cdot|\cdot)$, where

$$W^0 := \{x \in V \mid (x|y) = 0, \forall y \in W\}$$

is the orthogonal complement to W, as $G \cdot W^0 \subseteq W^0$

and $V = W \oplus W^0$.

Moreover, we also see that if we pick an orthonormal basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ for V, then upon identifying

$$\mathrm{Aut}_{\mathbb{C}}(V) \cong GL_n(\mathbb{C}),$$

we have $\rho(G) \subseteq U(n)$, where

$$U(n) := \{ M \in GL_n(\mathbb{C}) \mid M^* M = M M^* = I \}.$$

Here M^* is the conjugate transpose of M.

Given representations $\rho^{(1)}$ and $\rho^{(2)}$ of a given group G,

we may construct new ones by taking

$$\begin{aligned} G &\longrightarrow GL(V_1 \oplus V_2) \\ g &\longmapsto \rho^{(1)}_g \oplus \rho^{(2)}_g \end{aligned}$$

and their tensor product rep's

$$\begin{aligned} G &\longrightarrow GL(V_1 \otimes V_2) \\ g &\longmapsto \rho^{(1)}_g \otimes \rho^{(2)}_g \end{aligned}$$

More explicitly, if

$$\rho_g^{(1)}(e_{j_1}) = \sum_{i_1} r_{i_1 j_1}(g)\, e_{i_1}$$

$$\rho_g^{(2)}(e_{j_2}) = \sum_{i_2} r_{i_2 j_2}(g)\, e_{i_2}$$

then

$$\left(\rho_g^{(1)} \otimes \rho_g^{(2)}\right)(e_{j_1} \otimes e_{j_2}) = \sum_{i_1, i_2} r_{i_1 j_1}(g)\, r_{i_2 j_2}(g)\, e_{i_1} \otimes e_{i_2}$$