Lecture 22 The Schur orthogonality relations

Given a linear representation ρ of a finite group G in a vector space V over a field k, the character χ_ρ of ρ is

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[rrr, bend right, "\chi_\rho"'] & \mathrm{Aut}_k(V) \arrow[r, phantom, "\overset{\mathcal{B}}{\cong}"] & GL_n(k) \arrow[r, "\mathrm{Tr}"] & k \end{tikzcd}$$

where $\mathrm{Tr}(M) = \sum_{i=1}^{n} a_{ii}$, with $M = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$.

From linear algebra we know that

$$\mathrm{Tr}(\rho_g) = \sum_{i=1}^{n} \lambda_i ,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of ρ_g with multiplicities. Hence χ_g does not depend on the choice of basis $\mathcal{B}$ and it is thus well-defined. From now on we shall assume that $k = \mathbb{C}$.

<u>Prop'n 1</u> Notation as above

(i) $\chi(1) = n$

(ii) $\chi(g^{-1}) = \overline{\chi}(g)$, $\forall\, g \in G$

(iii) $\chi(hgh^{-1}) = \chi(g)$, $\forall\, g, h \in G$.

Proof

(i) Note that $\rho_1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, so $\chi(1) = n$.

(ii) As $g^n = 1$, for some $n \in \{1, 2, \dots\}$, then $\rho_g^n = I$ and thus

$$\lambda_i^n = 1$$

for each eigenvalue λ_i of ρ_g. Thus each λ_i is a root of unity, so

$$\overline{\chi}(g) = \overline{\mathrm{Tr}(\rho_g)} = \sum_{i=1}^n \overline{\lambda}_i = \sum_{i=1}^n \lambda_i^{-1} = \mathrm{Tr}(\rho_g^{-1}) = \chi(g^{-1}).$$

(iii) Note $\chi(hgh^{-1}) = \chi(g) \iff \chi(uv) = \chi(vu)$;

just put $u := hg$ and $v := h^{-1}$! But from linear algebra

we know that

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA),$$

for all $A, B \in M_n(k)$ $\square$

<u>Remark</u> Property (iii) says that χ_ρ is a class function.

We'll show that every class function is a sum of characters.

Prop'n 2 If $\rho^{(i)} : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V_i)$ are rep'n's, $i = 1, 2$, then

(i) $\chi_{\rho^{(1)} \oplus \rho^{(2)}} = \chi_{\rho^{(1)}} + \chi_{\rho^{(2)}}$

(ii) $\chi_{\rho^{(1)} \otimes \rho^{(2)}} = \chi_{\rho^{(1)}} \cdot \chi_{\rho^{(2)}}$

Proof

(i): ✓

(ii): We have $\chi_1(g) = \sum_{i_1} r_{i_1 i_1}(g)$ & $\chi_2(g) = \sum_{i_2} r_{i_2 i_2}(g)$,

so $\chi_{\rho^{(1)} \otimes \rho^{(2)}}(g) = \sum_{i_1 i_2} r_{i_1 i_1}(g) \cdot r_{i_2 i_2}(g) = (\chi_{\rho^{(1)}} \cdot \chi_{\rho^{(2)}})(g)$ □

<u>Prop'n 3</u> (Schur lemma) Let $\rho^{(i)} : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V_i)$ be irred'l rep'ns, $i = 1, 2$, and let

$$V_1 \xrightarrow{f} V_2$$

be a $\mathbb{C}$-linear mapping of G-sets. Then

(1) if $\rho^{(1)}$ and $\rho^{(2)}$ are not isomorphic, then $f = 0$;

(2) if $\rho^{(1)} = \rho^{(2)}$ then f is just multiplication by a scalar.

Proof

(1): Suppose $f \neq 0$. Let $W_1 := \ker(f)$ and note that $\forall x \in W_1$: $f(g \cdot x) = g \cdot f(x) = 0$, so W_1 is subrep'n of V_1. But V_1 is irred'l, so either

$$W_1 = V_1 \quad \text{or} \quad W_1 = 0.$$

The first possibility contradicts $f \neq 0$, so $W_1 = 0$, i.e. f is injective. Similarly we see that $\mathrm{Im}(f) = W_2$, so we have shown that f is an isomorphism.

(2): Suppose that $\rho^{(1)} = \rho^{(2)}$. Let λ be an eigenvalue of f (which must exist as $\mathbb{C}$ is algebraically closed).

Let $f' := f - \lambda \cdot 1_V$, $V := V_1 = V_2$. Note that

$$W := \ker(f') \neq \{0\},$$

which is a subrep'n of V (as $\forall g \in G, v \in V$: $f'(g \cdot v) = g \cdot f'(v)$). But V is irreducible, so we have $W = V$, i.e $f' = 0$. Thus $f = \lambda 1_V$ $\square$

For complex-valued functions $G \underset{\psi}{\overset{\phi}{\rightrightarrows}} \mathbb{C}$ we put

$$(\phi \mid \psi) := \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)} ,$$

which is clearly a scalar product. From Schur's lemma we get the following.

Thm If $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is irreducible then

$$(\chi_\rho \mid \chi_\rho) = 1;$$

if $\rho^{(i)} : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V_i)$ are irred'l and $\rho^{(1)} \not\cong \rho^{(2)}$, then

$$(\chi_{\rho^{(1)}} \mid \chi_{\rho^{(2)}}) = 0$$

<u>Thm</u> Let $V = V_1 \oplus \cdots \oplus V_k$ be a decomposition of ρ into irred'l subrep'ns $V_1, \ldots, V_k$ and pick any irred'l rep'n

$$\rho' : H \longrightarrow \mathrm{Aut}_{\mathbb{C}}(W)$$

Then the # of $V_i \cong W$ is $(\chi_\rho | \chi_{\rho'})$.