Irreducible representations of the symetric group

based on the article by McNamara, R.

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Basic of Representation Theory

Definition 1.1 A **General Linear Group** over $\mathbb{C}$, $GL(n, \mathbb{C})$ is the set of all nxn invertible matrices over $\mathbb{C}$. If V is a vector space then $GL(V)$ is the set of all automorphisms of V.

Definition 1.2 A **representation** φ of a group G is a homomorphism $\varphi : G \to GL(n, \mathbb{C})$.

Definition 1.3 A G-**module** V is a vector space for which there exists a homomorphism $\varphi : G \to GL(V)$.

The representation φ induces a multiplication G in V. Let $g, h, e \in G$; $v, w \in V$; $\alpha \in \mathbb{C}$ and denoted $\varphi(g)$ as g, then define the operation $g \star v = \varphi(g)v$ and

(1) $g \star (v + w) = g \star v + g \star w$,

(2) $\alpha(g \star v) = g \star (\alpha v)$,

(3) $(gh) \star v = g \star h \star v$,

(4) $e \star v = v$.

Definition 1.4 The **group algebra**, denoted $\mathbb{C}[G]$, is the set of all linear combinations of elements of G.

$|G| = n$

$$C[G] = \left\{ \sum_{g \in G} c_g g | c_g \in \mathbb{C} \right\}$$ and

$h \star (c_1 g_1 + \cdots + c_n g_n) =$

$c_1(hg) + \cdots + c_n(hg)$

$$h \star \left(\sum_{g \in G} c_g (hg) \right) = \sum_{g \in G} c_g (hg)$$ for all $h \in G$.

Definition 1.5 Reducible and Irreducible. If a module has a non-trivial, invariant proper subspace, then it is said to be reducible. A module that is not reducible is said to be irreducible.

Definition 1.6 Trivial representation The unit or trivial representation of G is the representation $\varphi : G \to GL(\mathbb{C})$ such that $\varphi(g) = [1]$ for every $g \in G$.

Definition 1.7 Alternating representation If G has a subgroup H with index 2, then we can define the alternating representation associated to the pair (G, H) as the representation $\varphi : G \to GL(\mathbb{C})$ such that $\varphi(g) = [1]$ if $g \in H$ and $\varphi(g) = [-1]$ otherwise.

Definition 1.8 Standard representation The standard representation of a symmetric group on a finite set of degree n is an irreducible representation of degree $n-1$.

The group S_n acts on $\mathbb{C}^n$ by permuting basis vectors. This representation has a 1-dimensional invariant subspace, spanned by the vector $e_1 + e_2 + \ldots + e_n$, which is the trivial representation. A complementary subspace to this is

$$V = \{a_1 e_1 + \ldots + a_n e_n | a_1 + \ldots + a_n = 0\}$$

$a_1 + \cdots + a_{n-1} = -a_n$

where V is the standard representation of S_n.

Example

The representation of S_3. $\varphi : S_3 \to \mathbb{C}^*$

1) The trivial representation that sends every element to [1].

2) The sign representation φ, that sends every element to its sign

$$(12) \mapsto [-1], \qquad (132) \mapsto [1]$$

3) The defining representation, φ' that permutes the columns of the 3x3 identity matrix:

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (132) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, etc.$$

And the corresponding **modules**.

$\varphi \longleftrightarrow [1]$

1) The one-dimensional vector space $\mathbb{C}$. Since for all $c \in \mathbb{C}$,

$g \star c = c$ $\qquad \varphi(g)c = [1]c = c$

2) The one-dimensional vector space $\mathbb{C}$. Since for all $c \in \mathbb{C}$,

$= \varphi(g)\, c = g \star c = c$ if g is an even permutation,

$\varphi(g) = g \star c = -c$ if g is an odd permutation.

3) The three-dimensional vector space $\mathbb{C}^3$. Where the action by a group element just permutes the coordinates of the vector. For example,

$\varphi : G \to GL_3(\mathbb{C})$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = (12) \star \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Notice that modules from **1)** and **2)** are irreducible since they are one-dimensional.

While the module from **3)** is reducible (by Definition 1.8) since we have a invariant subspace $W = \textit{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$.

By Definition 1.8 the complementary subspace of W $\underline{\underline{\mathbb{C}^3 = V \oplus W}}$

$$V = \{a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} | a_1 + a_2 + a_3 = 0\} \to \dim V = 2$$

is the modulo for the irreducible standard representation

$$\to \left[\begin{array}{c|c} A(g) & C(g) \\ \hline 0 & B(g) \end{array}\right]$$

Then,

$$(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (12) \mapsto \left[\begin{array}{c|cc} 1 & a & b \\ \hline 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right], \quad (13) \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$(23) \mapsto \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad (123) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad (132) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

Maschke's Theorem

Theorem 1.9 (Maschke's Theorem).
Given a group G and a non-zero module V,

$$V = W^{(1)} \oplus ... \oplus W^{(k)},$$

where $W^{(i)}$ are irreducible representations.

Proof

Proposition 1.10 Let V be a G-module, W a submodule, and $\langle \cdot, \cdot \rangle$ an inner product invariant under the action of G. Then $W^{\perp}$ is also a G-submodule.

We will induct on $d =$dim$V$. If $d = 1$, then V we are done. Now suppose that $d > 1$. If V is reducible, then V has a nontrivial G-submodule, W. (We will construct a submodule complement for w)

Pick any basis $B = \{v_1, v_2, ..., v_d\}$ for V. Consider the unique inner product that satisfies

$$\langle v_i, v_j \rangle = \delta_{i,j}$$

for elements of B. This product may not be G-invariant, but we can come up with another one that is. For any $v, w \in V$ we let the inner product

$$\langle v, w \rangle' = \sum_{g \in G} \langle gv, gw \rangle.$$

To show that it is G-invariant under $\langle \cdot, \cdot \rangle'$, we wish to prove

$$\langle hv, hw \rangle' = \langle v, w \rangle'$$

for all $h \in G$ and $v, w \in V$.

But

$$\langle hv, hw\rangle' = \sum_{g\in G} \langle ghv, ghw\rangle = \sum_{f\in G} \langle fv, fw\rangle = \langle v, w\rangle'$$

as desired.

If we let

$$W^\perp = \{v \in V : \langle v, w\rangle' = 0\},$$

then by Lemma 1.10 we have that $W^\perp$ is a G-submodule of V with

$$V = W \oplus W^\perp.$$

Now we can apply induction to W and $W^\perp$ to write each as a direct sum of irreducibles. Putting these two decompositions together, we see that V has the desired form. ■

Lemma 1.11 (Schur's Lemma). Let V and W be two irreducible G-modules and let Φ be a homomorphism that preserves g action, $\Phi : V \to W$ and $\Phi(g \star v) = g \star \Phi(v)$. Then either Φ is an isomorphism or Φ is the trivial map.

Proof. Consider the kernel of Φ, which is a vector space. Then, by definition of Φ, for all $v \in \ker \Phi$, the vector $\Phi(g \star v) = g \star \Phi(v) = g \star 0 = 0$.Thus, $g \star v \in \ker \Phi$, for all $v \in V$. Therefore, the kernel is a vector space invariant under action by g. But, irreducible modules do not have nontrivial subspaces, so either $\ker \Phi = \{0\}$, in which case Φ is an isomorphism, or $\ker \Phi = V$, in which case Φ is the trivial map.

Characters and Conjugacy Classes

In this section we will show that the number of conjugacy classes is an upper bound.

Definition 2.1 The **character** of a group element g with respect to some representation φ, denoted $\chi(g)$ and later $\theta(g)$, is just the trace of the matrix for $g : \chi(g) = tr(\varphi(g))$.

Proposition 2.2 For any character χ and group element g, $\chi(g)^* = \chi(g^{-1})$ where $*$ denotes complex conjugation.

Proposition 2.3 If g and h are group elements in the same conjugacy class K, then g and h have the same character: $\chi(g) = \chi(h)$.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2 ortonormal

Proof 2.2 By picking the orthogonal basis of the V, where V is the G-modulo, we obtain a matrix representation Y for χ, where each $Y(g)$ is unitary, i.e.
$Y(g^{-1}) = Y(g)^{-1} = (Y(g)^*)^t$ then

$$\chi(g)^* = \mathrm{tr}\, Y(g)^* = \mathrm{tr}\, Y(g^{-1})^t = \mathrm{tr}\, Y(g^{-1}) = \chi(g^{-1}).\blacksquare$$

because h and g are in the same conjugacy class

Proof 2.3 By hypothesis $g = khk^{-1}$ and definition 2.1,

$$\chi(g) = \mathrm{tr}\varphi(g) = \mathrm{tr}\varphi(k)\varphi(h)\varphi(k)^{-1} = \mathrm{tr}\varphi(h) = \chi(h).\blacksquare$$

Continue with Example of S_3.The identity element has trace 3 and is in a conjugacy class all to itself.The two-cycles (12), (13), and (23) all have character 1 and are in the same conjugacy class; and the three cycles (123) and (132) both have character 0 and are also in the same conjugacy class. Denote these conjugacy classes with K(1);K(2), and K(3), respectively.
We can now compute the character of each conjugacy class for every irreducible representation in the next table:

		K^1	K^2	K^3
trivial	χ^1	1	1	1
Alternante	χ^2	1	-1	1
Standard	χ^3	2	0	-1

$K^1 = \{ e \}$

$K^2 = \{ (12), (13, (32) \}$

$K^3 = \{ (123), (132) \}$

We will define a product operation that ensures that such tables have orthogonal rows. We define our inner product for the characters as

$$\begin{aligned}\langle \chi, \theta \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot \theta(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot \theta(g)^* \\ &= \frac{1}{|G|} \sum_{K} |K| \cdot \chi(K) \cdot \theta(K)^*\end{aligned} \tag{1}$$

Let φ and ψ be inequivalent, irreducible representations of a group G. Call their corresponding characters χ and θ respectively. Later, we prove $\langle \chi, \theta \rangle = 0$

Suppose φ has dimension m and ψ has dimension n. In order to define a homomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ for every pair i, j where $i \leq m$ and $j \leq n$, we first define the mxn matrix $E_{i,j}$ where the $i.j^{th}$ entry is 1 and all the other entries are 0. Then we define Φ as left multiplication by a matrix $F_{i.j}$

$$F_{i,j} = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \cdot E_{i,j} \psi(g^{-1})$$

$E_{1,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\Phi = F_{i,j} \cdot v$$

Now we want to prove that Φ is an homomorphism.

$\Phi : \mathbb{C}^n \to \mathbb{C}^m$

ψ degree n

φ degree m

$$\Phi(\overbrace{h \star v}^{\mathbb{C}^n}) = \overbrace{h \star \Phi(v)}^{\mathbb{C}^m} \Leftrightarrow$$

$$\Phi(\psi(h) \cdot v) = \varphi(h) \cdot \Phi(v) \Leftrightarrow$$

$$F_{i,j} \cdot \psi(h) \cdot v = \varphi(h) \cdot F_{i,j} \cdot v \Leftrightarrow$$

$$F_{i,j} \cdot \psi(h) = \varphi(h) \cdot F_{i,j} \Leftrightarrow$$

$$F_{i,j} = \varphi(h) \cdot F_{i,j} \cdot \psi(h^{-1})$$

So we just have to prove $F_{i,j} = \varphi(h) \cdot F_{i,j} \cdot \psi(h^{-1})$. Notice that $G = hG$ for some $h \in G$, specifically $g = hg$, then

$g = hg$

$$
\begin{aligned}
F_{i,j} &= \frac{1}{|G|} \sum_{g \in G} \varphi(g) \cdot E_{i,j} \cdot \psi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \varphi(hg) \cdot E_{i,j} \cdot \psi((hg)^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \varphi(h)\varphi(g) \cdot E_{i,j} \cdot \psi(g^{-1})\psi(h^{-1}) \\
&= \varphi(h) \cdot F_{i,j} \cdot \psi(h^{-1})
\end{aligned}
$$

So Φ is an homomorphism of irreducible representations. By Schur's lemma, Φ is either an isomorphism or the zero map.

$\varphi \neq \psi$

Since the representations are inequivalent, Φ is the zero map and $F_{i,j}$ is the zero matrix for all i,j, in particular the i,j^{th} entry of $F_{i,j}$ is zero, so

$$\begin{bmatrix} a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} \\ 0 & 0 & 0 \end{bmatrix}$$

$$0 = i,j^{th} \text{ entry of } F_{i,j} \tag{2}$$

$$= \frac{1}{|G|}\sum_{g\in G} \varphi(g)_{i,i}\cdot\psi(g^{-1})_{j,j}. \tag{3}$$

$$\underset{\varphi(g)}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \underset{E_{ij}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} \underset{\psi(g^{-1})}{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}$$

$$\begin{bmatrix} b_{31} & b_{32} & b_{33} \\ 0 & 0 & 0 \end{bmatrix}$$

Lemma 2.4. The inner product of any two characters associated with two inequivalent, irreducible representations is 0.

Proof.

$$\chi(g)\theta(g^{-1}) = \sum_{i\leq m}\varphi(g)_{i,i}\cdot\sum_{j\leq n}\psi(g^{-1})_{j,j} = \sum_{i,j}\varphi(g)_{i,i}\cdot\psi(g^{-1})_{j,j}$$

where ψ is the representation corresponding to θ. By combining Equation (1) and Equation (3), we find,

$$\langle\chi,\theta\rangle = \frac{1}{|G|}\sum_{g\in G}\chi(g)\cdot\theta(g^{-1}) = \frac{1}{|G|}\sum_{i,j}\sum_{g\in G}\varphi(g)_{i,i}\cdot\psi(g)_{j,j} = 0. \blacksquare$$

Lemma 2.5 Let G be a group with K conjugacy classes denoted $K^{(1)}, \dots, K^{(k)}$ and let φ and ψ be inequivalent, irreducible representations with associated characters χ and θ. Suppose v is a vector where the i^{th} entry of v is given by $\chi(K^{(i)})$ and similarly the $\theta(K^{(j)})$. So, is the j^{th} entry of some vector w. Then v and w are linearly independent.

Proof. To prove v and w are linearly independent, we temporarily define a new inner product operation on two vectors x and y of length k where xi denotes the i^{th} entry of x.

$$\langle x, y \rangle = \frac{1}{|G|} \cdot \left(|K^{(1)}| \cdot x_1 \cdot y_1 + \dots + |K^{(k)}| \cdot x_k \cdot y_k \right).$$

The reader can check that this, in fact, defines an inner product operation. Notice that in our case,

$$\langle w, v \rangle = \langle \chi, \theta \rangle = 0$$

.

Consider a linear combination of v and w that gives 0,

$$c_1 v + c_2 w = 0.$$

To show that $c_1 = 0$, we simply take the inner product of both sides and simplify

$$\begin{aligned} \langle c_1 v + c_2 w, v \rangle &= \langle 0, v \rangle \\ c_1 \langle v, v \rangle + c_2 \langle w, v \rangle &= \langle 0, v \rangle \\ c_1 \langle v, v \rangle + 0 &= 0 \\ \langle v, v \rangle &\neq 0 \\ c_1 &= 0. \end{aligned}$$

A similar argument will show that $c_2 = 0$. $\blacksquare$

Theorem 2.6 The number of irreducible representations is at most the number of conjugacy classes.

Proof. Suppose a group G has k conjugacy classes. Since the set of inequivalent, irreducible representation correspond to linearly independent vectors of length k, there can be at most k of them.

#of irreducible representations = #of rows

=

= #of linearly independent rows

= #of linearly independent columns

$\leq$ #of columns

= # conjugacy classes

Therefore, the number of conjugacy classes is an upper bound on the number of inequivalent, irreducible representations. ■

Specht Modules

In this section, we will build a set of modules, known as Specht modules, for each conjugacy class of the symmetric group.

Definition 3.1 A partition of a positive integer n is a sequence of positive integers, $\lambda = (\lambda_1, \lambda_2, \ldots)$ in non-increasing order that sum to n.

- $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$

The partitions of n are canonically associated with the cycle shapes of S_n.

- $\lambda \geq \lambda_2 \geq \ldots$

For example, the partition $(3, 2, 2, 1)$ of 8 is associated with the permutations on 8 letters with one three-cycle, two two-cycles and one one-cycle.

Lemma 3.1 The conjugacy classes of S_n are determined entirely by cycle shape.

Proof. Let p and q be permutations. Let us conjugate q by p. We will first show that to find pqp^{-1}, we just have to apply p point-wise to the cycles of q. Let q send some n to $q(n)$. Then, $(pqp^{-1})(p(n)) = (pq)(p^{-1}p)(n) = p(q(n))$. Since p is applied point-wise to q, the cycle shape is retained. We can also conjugate two permutations of the same cycle shape into one another. If we want to conjugate q into q', we just find some p that maps cycle to cycle. Thus, q and q' are conjugate by this p. ■

$pqp^{-1} = q'$

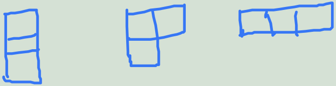
$q' = p^{-1}qp$

Example

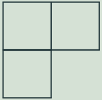
Consider two elements of S_3, (12) and (13). Now if the proposition is true, then these two elements are conjugate. For example, if we conjugate by (23), we get $(23) \circ (12) \circ (23) = (13)$. Similarly, the proposition predicts that any conjugation will preserve cycle shape. For example, $\underbrace{(123)}_{\rho} \circ (12) \circ \underbrace{(132)}_{\rho^{-1}} = (23)$.

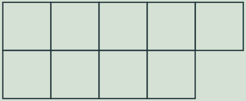
A Young diagram is just an array of boxes, with nonincreasing row length. The length of each row represents the size of one cycle.

$(1,1,1)$ $(2,1)$ (3) ← particiones de 3



Example

one two-cycle, one one-cycle $= (2,1) =$ 

one five-cycle, one four-cycle $= (5,4) =$ 

We get a filling by filling in each box with a number. A **standard filling** satisfies two conditions: the entries are ordered in decreasing value along the rows and down the columns, and it represents a permutation i.e. the numbers 1 through n are used precisely once. The resulting array is a standard Young tableau.

Example

$$(12)(3) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$$(13578)(2469) = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 5 & 7 & 8 \\ \hline 2 & 4 & 6 & 9 & \multicolumn{1}{c}{} \\ \cline{1-4} \end{array}$$

We define the group action pointwise: for any permutation g, we just apply g individually to the entries of the tableau T. Let us do some more examples. Notice that we defined action by g to have the same effect as conjugation by g.

Example

$$(13)\star \overset{T}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = \overset{p}{(13)} \overset{\circ}{\circ} \overset{q}{(12)} \overset{\circ}{\circ} \overset{p^{-1}}{(13)}$$

$$(1589)\star \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 5 & 7 & 8 \\ \hline 2 & 4 & 6 & 9 & \multicolumn{1}{c}{} \\ \cline{1-4}\end{array} = \begin{array}{|c|c|c|c|c|}\hline 5 & 3 & 8 & 7 & 9 \\ \hline 2 & 4 & 6 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4}\end{array}$$

$$= \underset{p}{(1589)} \circ \underbrace{(13578)(2469)}_{q} \circ \underset{p^{-1}}{(1985)}$$

A tabloid is such an equivalence class of Young tableaux. A tabloid corresponding to T is denoted by $\{T\}$.

Let us say we have a Young tableau we define the equivalents Young tableaus as

(123)(45)

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \sim \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 5 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \sim \begin{array}{|c|c|c|}\hline 1 & 3 & 2 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \sim \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

(123)(45)

if we have two equivalent tableaux we get back a specific permutation that takes the first diagram into the second.

So, the row stabilizer for a tableau T, denoted $R(T)$, is the same as picking a permutation on the first set of letters, then one on the second set of letters and so on. $R(T) = S_{R_1} \times \cdots \times S_{R_\ell}$

The column stabilizer, denoted $C(T)$. We found elements of the row stabilizer by just rearranging the entries of the rows.

$(145)\ g * T$ $\qquad$ $C(T) = S_{C_1} \times \cdots \times S_{C_n}$

Let have the tableau $T' = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}$, then

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ c_1 & c_2 & c_3 \end{array}$$

$S_{C_1} \times S_{C_2} \times S_{C_3}$

- $R(g * T) = g\, R(T)\, g^{-1}$
- $C(g * T) = g\, C(T)\, g^{-1}$

$$R(T) = \{(1), (12), (13), (23), (45), (46), (56), (12)(45), (12)(46), (12)(56), (13)(45), (13)(46), (13)(56), (23)(45), (23)(46), (23)(56), (123), (132), (456), (465), (123)(456), (123)(465), (132)(456), (132)(465)\},$$

$$C(T) = \{(1), (14), (25), (36), (14)(25), (14)(36), (25)(36), (14)(25)(36)\}.$$

Let T be a Young tableau. We define the associated **polytabloid**, denoted by e_T as follows:

$$e_T = \sum_{\pi \in C(T)} sgn(\pi)\pi \star \{T\}.$$

$\boxed{1|2|3} \sim \boxed{3|2|1}$

e_T

Example $T = \{123\}$

$$e_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} = \left\{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right\}$$

$$e_{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}} = \operatorname{sgn}((1))(1) \star \left\{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\right\} + \operatorname{sgn}((15))(15) \star \left\{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\right\}$$

$$+ \operatorname{sgn}((15)(26))(15)(26) \star \left\{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\right\} + \operatorname{sgn}((26))(26) \star \left\{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\right\}$$

$$= \left\{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|c|}\hline 5&2&3&4\\\hline 1&6\\\cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|c|}\hline 5&6&3&4\\\hline 1&2\\\cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|c|}\hline 1&6&3&4\\\hline 5&2\\\cline{1-2}\end{array}\right\}$$

Proposition 3.2. For all $g \in G$, $g \star e_T = e_{g \star T}$. Next, we define the Specht module, usually denoted $S^{(\lambda)}$, where λ just specifies a cycle shape.

$C(g \star T) = g \circ C(T) \circ g^{-1}$

Proof.

$g \star \{T\} = \{g \star T\}$

$$\begin{aligned} e_{g\star T} &= \sum_{\pi \in C(g\star T)} sgn(\pi)\pi \star \{g \star T\} \\ &= \sum_{\pi \in g \circ C(T) \circ g^{-1}} sgn(\pi)\pi \star \{g \star T\} \\ &= \sum_{\pi' \in C(T)} sgn(g \circ \pi' \circ g^{-1}) g \circ \pi' \circ g^{-1} \star \{g \star T\} \\ &= g \star \sum_{\pi' \in C(T)} sgn(\pi')\pi' \star \{T\} \\ &= g \star e_T \end{aligned}$$

Definition 3.3. A Specht module is a module spanned by polytabloids e_T, where T is taken over all tableaux of shape λ i.e. $S^{(\lambda)} = \{c_1 e_{T_1} + c_2 e_{T_2} + c_3 e_{T_3} + \ldots | c_1, c_2, \ldots \in \mathbb{C}, T_1, T_2, \ldots$ are tableaux of shape λ.

The trick to figuring out what $\boxed{S^{(\lambda)}}$ looks like for $\lambda = (1,1,1)$ is to look at g action on a basis element e_T . For any group element g, we have the following equalities:

$T =$ 3! T $\underbrace{e_{T_j}}$ $\leftarrow$ 3! $\sum_{T \in C(T)}$

T_j $1 \leq j \leq 6$

$$
\begin{aligned}
g \star e_T &= g \star \sum_{\pi \in C(T)} sgn(\pi)\pi \star \{T\} \\
&= \sum_{\pi \in S(3)} sgn(\pi) \cdot g \circ \pi \star \{T\} \\
&= \sum_{\pi \in S(3)} sgn(g^{-1} g\pi) \cdot g \circ \pi \star \{T\} \\
&= \sum_{\pi \in S(3)} sgn(g^{-1}) sgn(g\pi) \cdot g \circ \pi \star \{T\} \\
&= \sum_{g\pi \in S(3)} sgn(g^{-1}) sgn(g\pi) \cdot g \circ \pi \star \{T\} \\
&= sgn(g^{-1}) e_T \\
&= sgn(g) e_T
\end{aligned}
$$

So $S^{(\lambda)}$ is the sign representation.

If we were to write out all the e_T , we would realize that for all T, $e_T = \pm e_{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}}$. Notice that, even though there were 3!

φ Trival

v G-module 1 dim

different tableaux, the resulting Specht module was still only one-dimensional.

Irreducibility of the Specht Modules

In this section, we will show that these Specht modules have no proper, nontrivial submodules invariant under the action of g.

Lemma 4.1 Let T and T' be two λ tableaux. Then $\sum\limits_{\in C(T')} sgn(\pi)\pi \star \{T\} = \pm e_{T'}$.

Proof. The argument is similar to the argument we used to prove $S^{(}\lambda)$ gives the sign representation when $\lambda = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$.

$$
\begin{aligned}
T &= g \star T' \text{for some } g \in G \\
\sum_{\pi \in C(T')} sgn(\pi)\pi \star \{T\} &= \sum_{\pi \in C(T')} sgn(\pi)\pi \star \{g \star T'\} \\
&= \sum_{\pi \in C(T')} sgn(\pi)\pi \circ \{g \star T'\} \\
&= \sum_{\pi \in C(T')} sgn(\pi \circ g) sgn(g^{-1}) \pi \circ g \star \{T'\} \\
&= sgn(g^{-1}) \sum_{\pi \in C(T')} sgn(\pi \circ g) \pi \circ g \star \{T'\} \\
&= sgn(g^{-1}) e_{T'} \\
&= \pm e_{T'}
\end{aligned}
$$

Lemma 4.2. Let W be a nontrivial subspace of $S^{(\lambda)}$ for some λ and let $w \in W$ such that $w \neq 0$. Choose some tableau T of shape λ. Then,

$$\sum_{\pi \in C(T)} sgn(\pi)\pi \star w = c \cdot e_T,$$

for some $c \in \mathbb{C}$. Since $w \neq 0$, $c \neq 0$.

Proof. Since $w \in S^{(\lambda)}$, it must be the sum of tabloids of shape λ indexed by i

$$w = \sum_i c_i \{T_i\}$$

By the previous lemma,

$$\sum_{\pi \in C(T)} sgn(\pi)\pi \star T_i = \pm e_T,$$

for all i. Then, we have the following equation:

$$\sum_{\pi \in C(T)} sgn(\pi)\pi \star w = \pm c_1 e_T \pm c_2 e_T \pm \ldots = c \cdot e_T, \quad c \in \mathbb{C}.$$

$e_T(\pm c_1 \pm c_2 \pm \ldots)$

Theorem 4.3 The Specht modules are irreducible.

Proof. Since W is a nontrivial subspace, it contains $w \neq 0$. By the previous lemma

$\in W$

$$\sum_{\pi\in C(T)} sgn(\pi)\pi\star w = c\cdot e_T.$$

for some $c \in C$. Since W is invariant under group action, $\pi\star w \in W$, for all $\pi \in C(T)$. Therefore, the linear combination,

$\in W$

$$\sum_{\pi\in C(T)} sgn(\pi)\pi\star w \in W$$

$g\star e_T = e_{g\star T}$
$= e_{T'}$

But, this implies that $c \cdot e_T \in W$ for some $c \neq 0$, and thus $e_T \in W$. However, since $g\star e_T = e_{g\star T}$, we can obtain any tableau T' from T by this g action. Since W is invariant under this g action, $e_{T'} \in W$ for all T' of shape λ. Therefore, $W = S^{(\lambda)}$.

Dimensions of the Irreducible Representations

In this section we prove that the polytabloids associated with the standard Young tableaux of shape λ form a basis for the Specht module $S^{(\lambda)}$

Theorem 5.1. The polytabloids associated with the standard Young tableaux form a basis for the corresponding Specht module: $S^{(\lambda)} = \{c_1 e_{T_1} + + c_k e_{T_k} | c_1, ..., c_k \in \mathbb{C}, T_1, ..., T_k\}$ are standard tableaux of shape λ.

We will prove this in parts; we will first show that the standard polytabloids are linearly independent and then show that they span the Specht modules. We begin with some machinery.

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Definition 5.2. We say $\{T\} < \{T'\}$ if there exists some i such that,

(1) for all $j > i$, j is in the same row of both $\{T\}$ and $\{T'\}$,

(2) i is in a higher row of $\{T\}$ than $\{T'\}$.

The ordering extends to tableaux just as one would expect: $T < T'$ if $\{T\} < \{T'\}$, where $T \leq T'$ and $T' \leq T$ implies $\{T\} = \{T'\}$.

Example: $\begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2} \end{array} < \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 1 & 3 \\ \cline{1-2} \end{array} < \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$ for the first inequality $i = 3$ and $i = 2$ for the second one.

Lemma 5.3. Suppose $\pi \in C(T)$ for some standard Young tableau T. Then,

$$\{\pi \star T\} \leq \{T\}.$$

Proof. Take the largest entry permuted by π and label it i. All $j > i$ are not permuted. Therefore, they are in the same row of T as $\pi \star T$. The columns are already in descending order. Therefore, any permutation must take the largest entry permuted to a higher row. Thus, $\{\pi \star T\} \leq \{T\}$.

Theorem 5.4 The polytabloids associated with the standard Young tableaux are linearly independent.

Proof Suppose we apply this ordering to the standard tableaux, $T_1 < T_2 < \ldots < T_k$ and that there is a linear combination of the associated polytabloids that gives 0,

$$c_1 e_{T_1} + c_2 e_{T_2} + \ldots + c_k e_{T_k} = 0.$$

We will begin by showing that $c_k = 0$. When we expand e_{T_k}, we get $\{T_k\} \pm \ldots$
In order to cancel $\{T_k\}$, it must show up again later down in the line. But, the terms in any other polytabloid are of the form $\pi \star \{T_l\}$ for some $l < k$. Applying the previous lemma, $\pi \star \{T_l\} \leq \{T_l\} < \{T_k\}$, so we cannot possibly cancel $\{T_k\}$ with any other polytabloid. Thus, $c_k = 0$. We can apply an inductive argument to show that $c_i = 0$, for all $i \leq k$.

We will now demonstrate a combinatorial process known as the straightening algorithm, which takes any standard tableau T and writes the associated polytabloid e_T in terms of other polytabloids closer to a linear combination of polytabloids associated with standard Young tableaux.

(1)Take a tableau T. Order the columns in decreasing order, which only changes the sign of the final linear combination.

For instance if you have the tableau $T = \begin{array}{|c|c|c|c|}\hline 1 & 9 & 3 & 6 \\ \hline 4 & 2 & 5 & \\ \cline{1-3} 8 & 7 & & \\ \cline{1-2}\end{array}$, then

you would end up with $T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 7 & 5 & \\ \cline{1-3} 8 & 9 & & \\ \cline{1-2}\end{array}$

(2) If the tableau is not yet standard, then there must be two adjacent entries in the same row where the left is greater than the right. If there are more than two such entries, we can just apply the algorithm to the top-most and then left-most pair for consistency. If, for instance,

$$T = \begin{array}{|c|c|c|c|l} \hline 1 & 2 & 3 & 6 & \multicolumn{1}{|l}{} \\ \hline 4 & 7 & 5 & \multicolumn{1}{|l}{\to B} & \\ \cline{1-3} 8 & 9 & \multicolumn{1}{|l}{\to A} & & \\ \cline{1-2} \end{array}$$

we focus on the pair 7 and 5. We isolate all the entries below the left out-of-order entry and above the right out-of-order entry. For example, we isolate 3, 5, 7 and 9. We call the entries below the left-out-of-order entry A and the entries above the right out-of-order entry B.

(3) Calculate the Garnir element, denoted by $g_{A,B}$, which is just the signed sum of all permutations of the isolated entries that keep both subsets A and B without column ascent, $g_{A,B} = \sum_{\pi} sgn(\pi)\pi$. For instance, in the above example $g_{A,B} = (1) - (57) + (579) - (375) + (37)(59) - (3795)$.

Lemma 5.5. Let T be a tableau and $g_{A,B}$ be a corresponding Garnir element. Then, $g_{A,B} \star e_T = \sum_{\pi}(sgn(\pi)\pi \star e_T) = 0$.

Proof.

Consider $\sum_{\sigma \in S_{A \cup B}} sgn(\sigma)\sigma \star \{T\} = 0$ where $S_{A \cup B}$ is the permutations of A and B. For all σ, there exist two adjacent elements a and b, and a transposition of the two elements denoted (ab). Then, $(ab) \star (\sigma \star \{T\}) = \sigma \star \{T\}$ and since $\mathrm{sgn}((ab) \circ \sigma)(ab) \circ \sigma \star \{T\} = -\mathrm{sgn}(\sigma)(ab) \circ \sigma \star \{T\}$ and $\mathrm{sgn}(\sigma)\sigma \star \{T\} = \mathrm{sgn}(\sigma)(ab) \circ \sigma \star \{T\}$ are both terms in the summand, the whole expression cancels to 0.

Now factor out the elements of the column stabilizer,

$$\sum_{\sigma \in S_{A\cup B}} sgn(\sigma)\sigma \star \{T\} = \sum_{\pi} \sum_{\sigma \in C(T)} sgn(\pi \circ \sigma)\pi \circ \sigma \star \{T\}.$$

For the remaining sum, we only need to choose one representative for each possible composition of the columns. If we choose each so that the columns are in descending order, then we get back $g_{A,B} \star e_T$,

$$\begin{aligned} 0 = \sum_{\pi} \sum_{\sigma \in C(T)} sgn(\pi \circ \sigma)\pi \circ \sigma \star \{T\} &= \sum_{\pi} sgn(\pi)\pi \star \overbrace{\left(\sum_{\sigma \in C(T)} sgn(\sigma)\sigma \star \{T\} \right)}^{e_T} \\ &= g_{A,B} \star e_T \\ &= 0. \end{aligned}$$

Theorem 5.6. The standard polytabloids span the corresponding Specht module.

Proof.

If $g_{A,B} = (1) \pm \pi_1 \pm \pi_2 \pm \ldots \pm \pi_k$ then we can multiply by e_T on the right to get $g_{A,B} \star e_T = e_T \pm \pi_1 \star e_T \pm_2 \star e_T \pm \ldots \pm \pi_k \star e_T$. Therefore, by the previous proposition, $e_T = \pm \pi_1 \star e_T \pm \pi_2 \star e_T \pm \ldots \pm \pi_k \star e_T$ and we have e_T written in terms of other polytabloids to which we can reapply the algorithm. These other polytabloids are somehow "closer" to the polytabloids associated with the standard tableaux, which we could formalize by defining a partial ordering on the rows. By induction, e_T is spanned by the standard polytabloids.

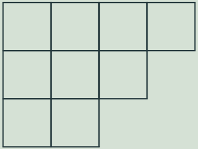
Theorem 5.1

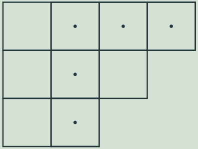
The polytabloids associated with the standard Young tableaux form a basis for the corresponding Specht module: $S^{(\lambda)} = \{c_1 e_{T_1} + + c_k e_{T_k} | c_1, ..., c_k \in C, T_1, ..., T_k\}$ are standard tableaux of shape λ.

Proof

By Theorem 5.4 and Theorem 5.6, the polytabloids associated with the standard Young tableaux a linearly independent and span, so they form a basis.
We have shown that the Specht modules give us all the irreducible representations of S_n. We conclude by calculating the dimensions of these representations

Definition 5.7 The hook-length of a given entry indexed i, j in a Young tableau T of shape λ, denoted $h_{i,j}$, is the number of entries to the right of i, j in row i plus the number of entries underneath i, j in column j plus 1.

Visually, if $\lambda =$ , then $h_{1,2}$ is computed by

counting dots in the following diagram: 

The tableau $\begin{array}{|c|c|c|c|}\hline 6 & 5 & 3 & 1 \\ \hline 4 & 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$ has $h_{i,j}$ as its i, j^{th} entry. Using this definition of the hook-length, we have the following formula, which counts the number of standard Young tableaux for a given shape λ of size n:

$$dim(S^{(\lambda)}) = \frac{n!}{\prod_{i,j} h_{i,j}}$$

The dimension of a representation corresponding to a partition λ is given by the standard Young tableaux of that shape, which we calculate using the last equation.

$|S_n| = \sum \dim^2(V_i)$

trivial φ^1 dim 1
alternante φ^2 dim 2
standard φ^3 dim 1

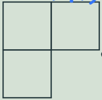
Example

Take, for instance, S_3. We have corresponding tableaux shapes

$\lambda = (3)$

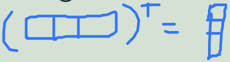
,

$\lambda = (2,1)$

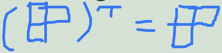


and

$\lambda = (1,1,1)$



. Then, by the hook-length



formula, we have that the dimension of the irreducible representations corresponding to these shapes are 1, 2, and 1, respectively, which agrees with the discussion so far.

$\lambda = (3)$

$h_{1,1} = 3$

$h_{1,2} = 2$

$h_{1,3} = 1$

$\frac{3!}{3^2 \cdot 2^2 \cdot 1^2} = 1$

$\lambda = (2,1)$

$h_{11} = 3$

$h_{12} = 1$

$h_{21} = 1$

$\frac{3!}{3 \cdot 1 \cdot 1} = 2$

$\lambda = (1,1,1)$

$h_{11} = 3$

$h_{21} = 2$

$h_{31} = 1$

$\frac{3!}{3 \cdot 2 \cdot 1} = 1$