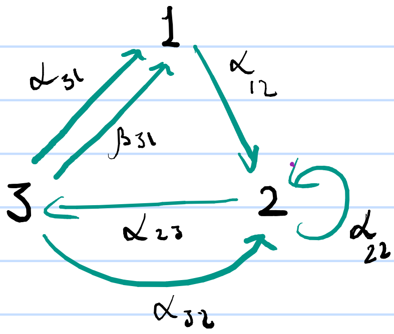
Quivers & Gabriel's Theorem

(Actually "Quiver Representations")

Preliminaries



Def. A quiver Q is a finite directed graph, denoted $Q = (Q_V, Q_A)$, where Q_V is the set of vertices, and Q_A is the set of arrows.
We will assume Q_V, Q_A finite.

$Q = (\{1,2,3\}, \{\alpha_{12}, \alpha_{22}, \alpha_{23}, \alpha_{32}, \alpha_{31}, \beta_{31}\})$

- For any $\alpha \in Q_A$, define:
  $s(\alpha) \in Q_V \leftarrow$ starting point of α,
  $t(\alpha) \in Q_V \leftarrow$ end point of α.

A non-trivial path in Q:

$$p = \alpha_r \cdots \alpha_2 \alpha_1 \qquad (\alpha_i \in Q_A)$$

s.t. $t(\alpha_i) = s(\alpha_{i+1}) \ \forall i = 1, \dots, r-1.$

$\hookrightarrow$ Denote $s(p) = s(\alpha_1)$; $t(p) = t(\alpha_r)$, length$(p) = r$.

* $\forall i \in Q_V$, we need a trivial path of length 0, e_i, and s.t. $s(e_i) = i = t(e_i)$.

* A path p is an oriented cycle if $s(p) = t(p)$ AND length$(p) > 0$.

Def. Let K be a field and Q a quiver. The Path Algebra KQ of Q over K has an underlying vector space w/ basis given by all paths in Q.

For any two paths $p = \alpha_r \cdots \alpha_1$, $q = \beta_s \cdots \beta_1$ in Q,

$$p \cdot q := \begin{cases} \alpha_r \cdots \alpha_1 \beta_s \cdots \beta_1 & \text{if } t(\beta_s) = s(\alpha_1) \\ 0 & \text{otherwise} \end{cases},$$

"concatenation of paths" (associative & distributive).

Define $\mathbb{1}_{KQ} = \sum_{i \in Q_V} e_i$; for any path p:

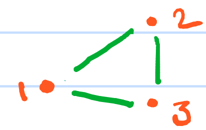
$$p \cdot \Big(\sum_{i \in Q_V} e_i\Big) = \sum_{i \in Q_V} p \cdot e_i = \underbrace{p \cdot e_{s(p)}}_{i = s(p)} = p = e_{t(p)} \cdot p = \Big(\sum_{i \in Q_V} e_i\Big) \cdot p$$

$\therefore \ \alpha \cdot \mathbb{1}_{KQ} = \alpha = \mathbb{1}_{KQ} \cdot \alpha \quad \forall \alpha \in Q_A$,

so indeed $\mathbb{1}_{KQ}$ is the identity element in KQ.

We wanna prove

Gabriel's Theorem: Assume Q is a quiver without oriented cycles, and K is a field. Then Q has finite representation type if and only if the underlying graph $\bar{Q}$ is the disjoint union of Dynkin diagrams of types A_n for $n \geq 1$, or D_n for $n \geq 4$, or E_6, E_7, E_8.

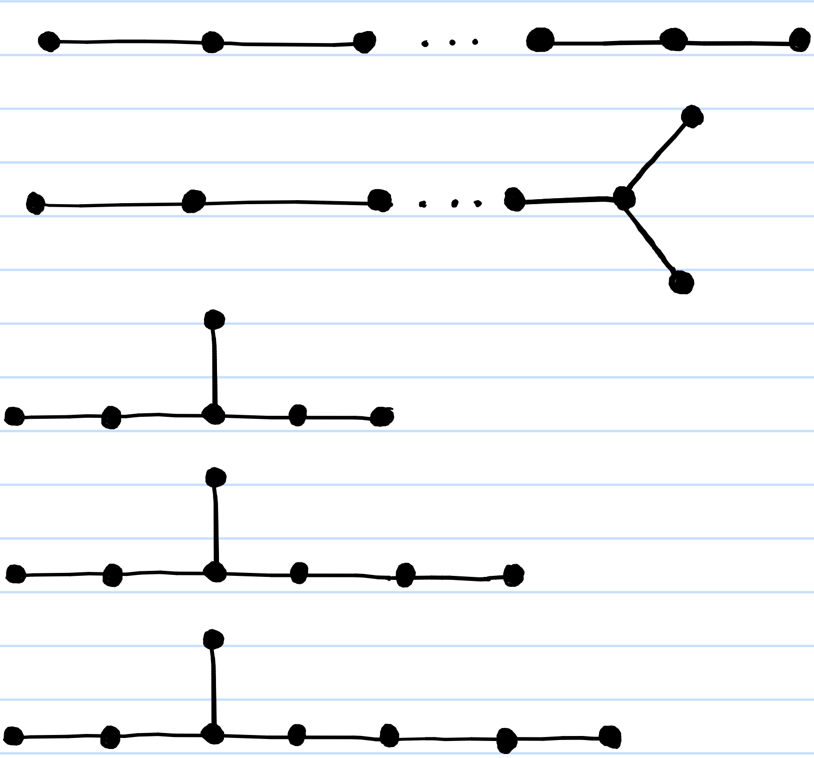


$\bar{Q}$

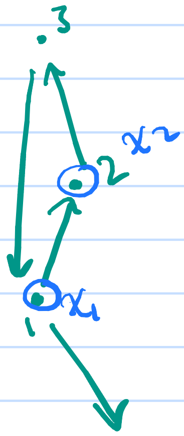
* Rep. type of Q does NOT depend on the orientation of the arrows, only on the underlying graph.

Fig. The Dynkin diagrams of type A, D, E.
(The index gives the number of vertices).

A_n, D_n, E_6, E_7, E_8



Lemma. Let Q be a quiver. If there is a path of length at least $|Q_v|$, $\Rightarrow$ there are *cyclic* paths.



Proof. Assume $\exists$ a path p in Q with $\text{length}(p) \geq |Q_v| := n$, say $p = \alpha_n \cdots \alpha_1$ for $\{\alpha_1, \dots, \alpha_n\} \subset Q_A$, and consider the vertices

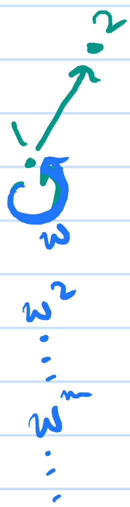
$$\{x_i = s(\alpha_i)\}_{1 \leq i \leq n} \cup \{x_{n+1} = t(\alpha_n)\} \subset Q_v .$$

There are $n+1$ vertices but $|Q_v| = n$! Thus $\exists\, i < j \leq n$ s.t. $x_i = x_j$. Then we have a path

$$w = \alpha_{j-1} \cdots \alpha_i$$

with $s(w) = t(w)$ and $\text{length}(w) > 0$.
$\therefore$ a cyclic path.

Prop. Let K be a field.
Then the path algebra KQ of a quiver Q is finite-dimensional if and only if Q does *not* contain any oriented cycles.



Proof. Assume Q finite, and consider an *oriented cycle* w in KQ. $\Rightarrow w^m$ is also a cycle for $m \geq 1$. Then, w gives rise to infinitely many paths

$$\{w, w^2, w^3, \dots\} \subset \mathcal{B}_{KQ},$$

where $\mathcal{B}_{KQ}$ is the basis for KQ. Hence KQ is infinite dimensional. By contrapositive this proves the "only if" implication.

Conversely, if Q does not have any cycles, then by the *Lemma* every path p in Q has finite length, $\text{length}(p) \leq |Q_v|$, so $\mathcal{B}_{KQ}$ consists of finitely many paths.

$\square$

Quiver Representations

* We want to represent vertices by vector spaces, and arrows by linear maps.

Def. Let $Q = (Q_V, Q_A)$ a quiver. A *representation* $\mathcal{V}$ of Q over a field K is a set of K-vector spaces $\{V(i) \mid i \in Q_V\}$ together with K-linear maps.

$$\{ V(\alpha) : V(i) \longrightarrow V(j) \mid i \xrightarrow{\alpha} j \in Q_A \}.$$

We write $\mathcal{V} = \left(\{V(i)\}_{i \in Q_V}, \{V(\alpha)\}_{\alpha \in Q_A} \right)$.

Example: Let Q be $1 \xrightarrow{\alpha} 2$.
Then a representation $\mathcal{V}$ consists of two K-vector spaces $\{V(1), V(2)\}$, and K-linear map $V(\alpha) : V(1) \longrightarrow V(2)$.

$$V(1) \xrightarrow{V(\alpha)} V(2)$$

* We can construct from this a *module* for the path algebra

$$KQ = \mathrm{span}\{e_1, e_2, \alpha\}.$$

Take as *underlying space*

$$V := V(1) \times V(2)$$

and let $e_i \in KQ$ act as projection $e_i : V \longrightarrow V(i)$ with $\mathrm{Ker}(e_1) = \{0\} \times V_2$, $\mathrm{Ker}(e_2) = V_1 \times \{0\}$.
Define the action of α by

$$\alpha((v_1, v_2)) = V(\alpha)(v_1) \qquad (v_1 \in V(1)).$$

Conversely, from a *KQ-module* V we obtain a *representation* of Q by

$$V(1) := e_1 V, \qquad V(2) = e_2 V,$$

and given $(v_1, v_2) \in V = (e_1 V) \times (e_2 V)$, then

$$V(\alpha) : e_1 V \longrightarrow e_2 V$$
$$v_1 \longmapsto \alpha((v_1, v_2)).$$

* This is true in general: rep's of a quiver Q over K define modules for KQ and viceversa.

Prop. Let K field and $Q = (Q_V, Q_A)$ quiver.
(a) Let $\mathcal{V} = (\{V(i)\}_{i \in Q_V}, \{V(\alpha)\}_{\alpha \in Q_A})$. Then

$$V := \prod_{i \in Q_V} V(i) = V(1) \times \cdots \times V(n), \quad n = |Q_V|.$$

becomes a KQ-module as follows:
let $v = (v_i)_{i \in Q_V} \in V$, and $p = \alpha_r \cdots \alpha_1 \in \mathcal{B}_{KQ}$ with $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_r)$. Then define the $|Q_V|$-tuple $p \cdot v$ by

$$(p \cdot v)_{\underset{\sim}{i}} = \delta_{i, s(p)} \cdot V(\alpha_r) \circ \cdots \circ V(\alpha_1)(v_{s(p)}).$$

In particular, if $r = 0$, $\Rightarrow e_i \cdot v = (0, \cdots, 0, v_i, 0, \cdots, 0)$, and this action is extended linearly to all of KQ.

(b) Let V be a KQ-module. For any vertex $i \in Q_V$ we set

$$V(i) = e_i V = \{ e_i \cdot v \mid v \in V \};$$

for any arrow $i \xrightarrow{\alpha} j$ in Q_A set

$$V(\alpha) : V(i) \longrightarrow V(j)$$
$$e_i \cdot v \longmapsto \alpha(e_i \cdot v) = \alpha \cdot v,$$

$\Rightarrow \mathcal{V} = (\{V(i)\}_{i \in Q_V}, \{V(\alpha)\}_{\alpha \in Q_A})$ is a representation of Q over K.

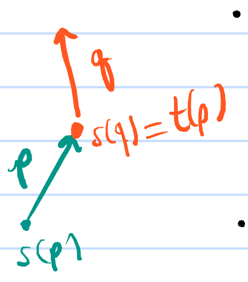
(c) Parts (a) and (b) are constructions inverse to each other.

Proof. (a) We will check that the module axioms are satisfied.

- Notice that each $V(i)$ is an abelian group $(V(i), +)$, so the product
$$V := \prod_{i \in Q_V} V(i) \;=\; V(1) \times \cdots \times V(n) \qquad (n = |Q_V|)$$
is an abelian group $(V, +_V)$ w/ $+_V$ defined componentwise.

- Let $p, q \in \mathcal{B}_{KQ}$ with $p = \alpha_r \cdots \alpha_1$. Note that by definition the KQ-action is distributive, hence
$$(p+q)\cdot v = p\cdot v + q \cdot v .$$

- Further, since $V(\alpha_i)$ is a K-linear map for each $\alpha_i \in Q_A$, if $v, w \in V$ then
$$\begin{aligned}(p\cdot(v+w))_i &= \delta_{i,s(p)} V(\alpha_r)\circ\cdots\circ V(\alpha_1)(v_{s(p)} + w_{s(p)}) \\ &= \delta_{i,s(p)}\big(V(\alpha_r)\circ\cdots\circ V(\alpha_1)(v_{s(p)}) \\ &\qquad + V(\alpha_r)\circ\cdots\circ V(\alpha_1)(w_{s(p)}) \big),\end{aligned}$$
so $p\cdot(v+w) = p\cdot v + p \cdot w$.



- Since $\forall\, p, q \in KQ$, $p\cdot q$ is concatenation, then $p\cdot(q\cdot v) = (pq)\cdot v \quad \forall v \in V$, and by linearity to all KQ.

- Finally, $\mathbb{1}_{KQ} = \sum_{i \in Q_V} e_i$, and $e_i : V \to V(i)$ acts by projection, so $\forall v \in V$:
$$\mathbb{1}_{KQ} \cdot v = \sum_{i \in Q_V} e_i \cdot v \;=\; v .$$

(b) We will show that the $V(i) = e_i V$ are K-vector spaces and that the maps $V(\alpha)$ are K-linear.

V is a module, so $\forall v, w \in V$ and $\lambda \in K$, we have
$$e_i\cdot v + e_i \cdot w = e_i\cdot(v+w) \in e_i V = V(i),$$
and
$$\begin{aligned}\lambda(e_i\cdot v) &= (\lambda \mathbb{1}_{KQ} e_i)\cdot v = (e_i \lambda \mathbb{1}_{KQ})\cdot v \\ &= e_i\cdot(\lambda v) \in e_i V := V(i).\end{aligned}$$

Further, V satisfies the K-module axioms, so they're satisfied in each $V(i)$ as well. Also, $(V, +_V)$ is an abelian group iff $(e_i V, +_i)$ is an abelian group for each $i \in Q_V$.

$\therefore$ Each $e_i V$ is a K-vector space.

Finally, $\forall \alpha \in Q_A$ with $i \xrightarrow{\alpha} j$, note $\alpha e_i = \alpha$, so indeed $V(\alpha) : e_i v \mapsto \alpha v$. Then $\forall\, \lambda, \mu \in K$; $v, w \in V(i)$:
$$\begin{aligned}V(\alpha)(\lambda v + \mu w) &= \alpha\cdot(\lambda v + \mu w) \\ &= (\alpha \lambda \mathbb{1}_{KQ})\cdot v + (\alpha \mu \mathbb{1}_{KQ})\cdot w \\ &= \lambda \mathbb{1}_{KQ}\cdot(\alpha\cdot v) + \mu \mathbb{1}_{KQ}(\alpha\cdot w) \\ &= \lambda V(\alpha)(v) + \mu V(\alpha)(w),\end{aligned}$$
so $V(\alpha)$ is a K-linear map.

(c) Exercise for the audience!

Def. Let $\mathcal{V} = (\{\mathcal{V}(i)\}_{i\in Q_V}, \{\mathcal{V}(\alpha)\}_{\alpha \in Q_A})$ be a representation of $Q = (Q_V, Q_A)$.

(a) A rep. $\mathcal{U} = (\{\mathcal{U}(i)\}_{i \in Q_V}, \{\mathcal{U}(\alpha)\}_{\alpha\in Q_A})$ of Q is a subrepresentation of $\mathcal{V}$ if:

(i) $\forall i \in Q_V$, $\mathcal{U}(i)$ is a subspace of $\mathcal{V}(i)$.
(ii) $\forall\, i \xrightarrow{\alpha} j$ in Q, the linear map $\mathcal{U}(\alpha) : \mathcal{U}(i) \to \mathcal{U}(j)$ is the restriction
$$\mathcal{U}(\alpha) = \mathcal{V}(\alpha)\big|_{\mathcal{U}(i)} .$$

(b) A non-zero rep. $\mathcal{S}$ of Q is simple if its only sub-reps. are 0 and $\mathcal{S}$.
$\nwarrow V(i) = 0.$

Def. Let Q quiver and K field.

(1) Let $\mathcal{M} = (\{\mathcal{M}(i)\}_{i\in Q_V}, \{\mathcal{M}(\alpha)\}_{\alpha \in Q_A})$ be a rep. of Q over K, and assume $\mathcal{U}, \mathcal{V}$ sub-reps. of $\mathcal{M}$.
$\Rightarrow$ $\mathcal{M}$ is the direct sum $\mathcal{U} \oplus \mathcal{V}$ if $\forall i \in Q_V$ we have
$$\mathcal{M}(i) = \mathcal{U}(i) \oplus \mathcal{V}(i)$$
as vector spaces.

(2) A non-zero rep. $\mathcal{M}$ of Q is indecomposable if it cannot be expressed as $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$ of non-zero sub-reps. $\mathcal{U}, \mathcal{V}$ of $\mathcal{M}$.

Quiver Reflections

Def. A K-algebra A has finite representation type if there are only finitely many finite-dimensional indecomposable A-modules, up to isomorphism.

Otherwise, A has infinite representation type.

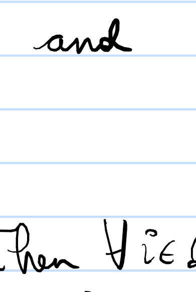
↳ Equivalently, A-modules of finite 'length'. Finite length $\Leftrightarrow$ Finite-dimensional.

* Isomorphic Algebras have the same rep. type

↳ Take $\Phi : A \xrightarrow{\sim} B$ w/ A, B K-algebras.
$\Rightarrow$ Every B-module becomes an A-module by setting
$$a \cdot m = \Phi(a)\, m$$
and every A-module becomes a B-module by
$$b \cdot m = \Phi^{-1}(b)\, m.$$
Note that this correspondence preserves dimensions and isomorphisms.

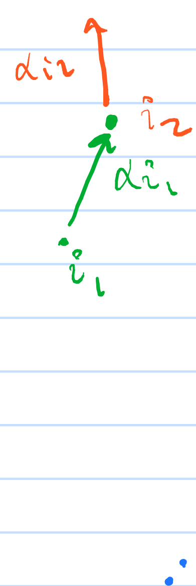
* Wanna prove that rep. type of a quiver does NOT depend on the direction of the arrows.
$\Rightarrow$ We need 'reflection' maps!

Def. Vertex j of Q is a sink if no arrow $\alpha \in Q_A$ has $s(\alpha) = j$. Vertex k of Q is a source if no arrow has $t(\alpha) = k$.



e.g. $1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4$

Lemma. If Q has no cycles, $\Rightarrow$ Q has a sink and a source.



Proof. Let $Q = (Q_V, Q_A)$ and $|Q_V| = n$.
Assume Q doesn't have a sink. Then $\forall i \in Q_V$ $\exists\, \alpha_i \in Q_A$ s.t. $s(\alpha_i) = i$. Then there is a sequence $i_1, \dots, i_n$ in Q_V s.t. $t(\alpha_{i_k}) = s(\alpha_{i_{k+1}})$ for $1 \le k < n$ so
$$p = \alpha_n \cdots \alpha_1$$
is a path w/ length$(p) = n$. Then, by the Lemma, $\exists\, i_k, i_\ell$ in this sequence s.t. $s(\alpha_{i_k}) = s(\alpha_{i_\ell})$, i.e. Q has a cycle.

Similarly, sup. Q has no sources. Then $\forall i \in Q_V$ $\exists\, \gamma_i \in Q_A$ s.t. $t(\gamma_i) = i$. Then similarly there is a path of length $|Q_V|$ and Q has a cycle.

By contrapositive, the Lemma is proved. $\square$

Def. Let $Q = (Q_V, Q_A)$ quiver and let $j \in Q_V$ be either a sink or a source. We define a new quiver $\sigma_j Q$ obtained by reversing all arrows adjacent to j, leaving the rest unchanged.

$\sigma_j Q$ is the reflection of Q at j.

* If j is a sink in Q, then j is a source in $\sigma_j Q$.

* We have $\sigma_j \sigma_j Q = Q$.

Ex. Consider

$$\begin{aligned}
Q &: 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4\\
\sigma_1 Q &: 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4\\
\sigma_2\sigma_1 Q &: 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4\\
\sigma_3\sigma_2\sigma_1 Q &: 1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4
\end{aligned}$$

* These are all quivers w/ underlying graph the Dynkin diagram of type A_4 (up to labeling of vertices).

This idea is more general.

Prop. Let Q, Q' quivers w/ the same underlying graph, which we assume to be a tree. ← NO cycles
$\Rightarrow$ $\exists\, i_1, \dots, i_r \in Q_V$ such that

(i) i_1 is a sink or source in Q.
(ii) $\forall j$ w/ $1 < j < r$, i_j is a sink or source in $\sigma_{i_{j-1}} \cdots \sigma_{i_1} Q$.
(iii) We have
$$Q' = \sigma_{i_r} \cdots \sigma_{i_1} Q.$$

Example: Let:

$$Q: \begin{tikzcd} & 5 \\ & 4 \arrow[u] & 2 \arrow[l] & 1 \arrow[l] \\ & 3 \arrow[u] \end{tikzcd} \qquad Q': \begin{tikzcd} 5 \arrow[d] \\ 4 \arrow[r] \arrow[d] & 2 \arrow[r] & 1 \\ 3 \end{tikzcd}$$

Consider the operation

$$Q \longrightarrow \hat{Q}: \begin{tikzcd} 4 & 2 \arrow[l] & 1 \arrow[l] \\ 3 \arrow[u] \end{tikzcd} \xrightarrow{\sigma_4\sigma_1} \hat{Q}': \begin{tikzcd} 4 \arrow[r] \arrow[d] & 2 \arrow[r] & 1 \\ 3 \end{tikzcd}$$

If we extend to the 5-vertex Q, we need to reflect 5 twice:

$$Q \xrightarrow{\sigma_1} \begin{tikzcd} 5 \\ 4 \arrow[u] & 2 \arrow[l] \arrow[r] & 1 \\ 3 \arrow[u] \end{tikzcd} \xrightarrow{\sigma_5} \begin{tikzcd} 5 \arrow[d] \\ 4 & 2 \arrow[l] \arrow[r] & 1 \\ 3 \arrow[u] \end{tikzcd}$$

$$\xrightarrow{\sigma_4} \begin{tikzcd} 5 \\ 4 \arrow[u] \arrow[r] \arrow[d] & 2 \arrow[r] & 1 \\ 3 \end{tikzcd} \xrightarrow{\sigma_5} \begin{tikzcd} 5 \arrow[d] \\ 4 \arrow[r] \arrow[d] & 2 \arrow[r] & 1 \\ 3 \end{tikzcd} : \; Q' = \sigma_5\sigma_4\sigma_5\sigma_1 Q.$$

$\Rightarrow$ Sequence $1, 5, 4, 5 \in Q_V$ s.t. the Proposition is satisfied.

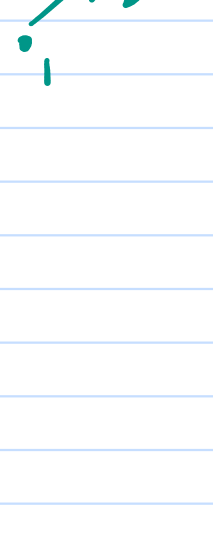
Proof. We proceed by induction on the number of vertices n. Let Γ be the underlying graph.

For $n=1$ or $n=2$ the statement is clear.

Assume the statement holds for for $|Q_V| \le n$ and $n \ge 3$. Since Γ is a tree, there must be a vertex, say n, which is adjacent to only one other vertex, say $n-1$.

Remove the vertex n and adjacent arrow (between n and $n-1$) from Q and Q'. This gives quivers $\hat{Q}, \hat{Q}'$, each w/ $n-1$ vertices and underlying graph $\hat{\Gamma}$, which is also a tree.

$\Rightarrow$ $\exists\, i_1, \dots, i_t \in \hat{Q}_V$ s.t. i_j is a sink or source in $\sigma_{j-1} \cdots \sigma_1 \hat{Q}$ for $1 < j < t$, and s.t. $\sigma_j \cdots \sigma_1 \hat{Q} = \hat{Q}'$.



Now we extend this to Q. We have 2 cases: either i_1 is a sink/source in Q itself and we set $Q^{(1)} := \sigma_{i_1} Q$.

Otherwise, i_1 is a sink or source in $\sigma_n Q$, and we set $Q^{(1)} := \sigma_{i_1} \sigma_n Q$.

We proceed in this way. If i_k is a sink/source in $Q^{(k-1)}$, set $Q^{(k)} = \sigma_{i_k} Q^{(k-1)}$. Otherwise, $i_k = n-1$ is a sink/source of $\sigma_n Q^{(k-1)}$, and we set $Q^{(k)} = \sigma_{i_k} \sigma_n Q^{(k-1)}$.

Repeat until you get $Q^{(t)}$ s.t. removing n and adjacent arrow produces $\hat{Q}'$. Then either

$$Q^{(t)} = Q' \quad \text{OR} \quad \sigma_n Q^{(t)} = Q'.$$

$\square$

The Reflection Σ_j^+ at a Sink.

* From a rep. $\mathcal{M}$ of Q we'll construct a rep. $\Sigma_j^+(\mathcal{M})$ of $\sigma_j Q$, where j is a sink in Q.

* The idea is to find a vector space $\mathcal{M}^+(j)$ and, for each arrow $\alpha_i : i \to j$, find a linear map $\mathcal{M}^+(\bar{\alpha}_i) : M^+(j) \to M(i)$, and leave the rest unchanged.

Def. Let j be a sink of Q. We label the distinct arrows ending at j by

$$\alpha_1, \alpha_2, \dots, \alpha_t, \quad \text{s.t. } \alpha_i : i \to j,$$

and we write $\bar{\alpha}_i : j \to i$ for the arrows of $(\sigma_j Q)_A$.

Ex. Let $t = 1$ and

$$Q: 1 \xrightarrow{\alpha_1} j \quad \text{and} \quad \sigma_j Q : 1 \xleftarrow{\bar{\alpha}_1} j.$$

Take rep. M of Q:
$$M(1) \xrightarrow{M(\alpha_1)} M(j)$$
and we want
$$M(1) \xleftarrow{M^+(\bar{\alpha}_1)} M^+(j)$$
using only info from M. Take
$$M^+(j) := Ker(M(\alpha_1))$$
and $M^+(\bar{\alpha}_1)$ to be the inclusion map. This defines a rep. $\Sigma_j^+(M) = (\{M(1), M^+(j)\}, \{M^+(\bar{\alpha}_1)\})$ of $\sigma_j Q$.

Ex. Now let $t = 3$. Take

$$Q: \begin{tikzcd} & 2 \arrow[d, "\alpha_2"] & \\ 1 \arrow[r, "\alpha_1"] & j & 3 \arrow[l, "\alpha_3"'] \end{tikzcd} \qquad \sigma_j Q: \begin{tikzcd} & 2 & \\ 1 & j \arrow[l, "\bar{\alpha}_1"'] \arrow[u, "\bar{\alpha}_2"'] \arrow[r, "\bar{\alpha}_3"] & 3 \end{tikzcd}$$

$\prod_{i=1}^{3} M(i)$ $\cup$

Sup. M is a rep. of Q, and we want $\Sigma_j^+(M)$ of $\sigma_j Q$.
Set $M^+(i) = M(i)$ for $i = 1, 2, 3$, and define

$$M^+(j) = \left\{ (m_1, m_2, m_3) \in \prod_{i=1}^{3} M(i) \;\middle|\; M(\alpha_1)(m_1) + M(\alpha_2)(m_2) + M(\alpha_3)(m_3) = 0 \right\}.$$

Also define
$$\begin{aligned} M^+(\bar{\alpha}_i) : M^+(j) &\longrightarrow M(i) \\ (m_1, m_2, m_3) &\longmapsto m_i \end{aligned} \qquad (i = 1, 2, 3).$$

Then $\Sigma_j^+(M) = (\{M(1), M(2), M(3), M^+(j)\}, \{M^+(\bar{\alpha}_1), M^+(\bar{\alpha}_2), M^+(\bar{\alpha}_3)\})$ is indeed a rep. of $\sigma_j Q$.

Def. Let Q quiver and assume j sink in Q. For a rep. M of Q, we define a rep. $\Sigma_j^+(M)$ of $\sigma_j Q$ as follows. Set

$$M(1) \xrightarrow{M(\alpha_1)} M(j) \xleftarrow{M(\alpha_2)} M(2)$$

$$\mathcal{M}^+(r) := \begin{cases} M(r) & (r \neq j) \\ \left\{ (m_1, \dots, m_t) \in \prod_{i=1}^{t} M(i) \;\middle|\; (M(\alpha_1), \dots, M(\alpha_t))(m_1, \dots, m_t) = 0 \right\} & (r = j). \end{cases}$$

$r \in Q_V$

If $\gamma \in Q_A$ s.t. $t(\gamma) \neq j$, set $M^+(\gamma) = M(\gamma)$.
Otherwise
$$\begin{aligned} M^+(\bar{\alpha}_i) : M^+(j) &\longrightarrow M^+(i) \\ (m_1, \dots, m_t) &\longmapsto m_i \end{aligned} \qquad , \; i = 1, \dots, t.$$

Then $\Sigma_j^+(M) = (\{M^+(i)\}_{i \in Q_V}, \{M^+(\bar{\alpha})\}_{\alpha \in Q_A})$ is a rep. of $\sigma_j Q$.

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* We want to compare the rep. types of Q and $\sigma_j Q$.
* Wanna keep track of direct sum decompositions.

Lemma. Let Q quiver and j sink in Q. Let M be a rep. of Q s.t. $M = X \oplus Y$ for sub-rep's X and Y of M.

$$\Rightarrow \Sigma_j^+(M) = \Sigma_j^+(X) \oplus \Sigma_j^+(Y).$$

Proof. "The proof is too technical"
— Erdmann & Holm.

Ex. Consider again $Q: 1 \xrightarrow{\alpha} 2$. We have rep's:

M	$\Sigma_2^+(M)$
$K \xrightarrow{M} 0$	$K \xleftarrow{id_K} K = Ker(M) := Ker(0)$
$K \xrightarrow{id_K} K$	$K \longleftarrow 0 := Ker(id_K)$
$0 \xrightarrow{0} K$	$0 \longleftarrow 0$

* We will formalize this notion in a bit.

Def. For each $j \in Q_V$, we have the simple representation S_j of Q over K given by

$$S_j(i) = \begin{cases} K & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \begin{tikzcd} 0 \arrow[r] & K_j & \\ & 0 \arrow[u] & 0 \arrow[l] \end{tikzcd} \quad S_j$$

and $S_j(\alpha) = 0 \quad \forall \alpha \in Q_A$.

Def. Let $Q = (Q_V, Q_A)$ and M, N rep's of Q over K. A homomorphism of representations $\varphi : M \to N$ is a tuple $(\varphi_i)_{i \in Q_V}$ of K-linear maps $\varphi_i : M(i) \to N(i)$ for each $i \in Q_V$, such that for each $i \xrightarrow{\alpha} j$ in Q_A, then

$(\varphi_{i_1}, \dots, \varphi_{i_n})$

$$\begin{tikzcd} M(i) \arrow[r, "M(\alpha)"] \arrow[d, "\varphi_i"'] \arrow[dr, phantom, "\circlearrowright"] & M(j) \arrow[d, "\varphi_j"] \\ N(i) \arrow[r, "N(\alpha)"] & N(j) \end{tikzcd}$$

commutes, i.e. $\varphi_j \circ M(\alpha) = N(\alpha) \circ \varphi_i$.

Prop. Let Q quiver and assume j sink in Q. Let M be a rep. of Q.

(a) $\Sigma_j^+(M) = 0$ iff $M(r) = 0 \quad \forall r \neq j$.

(b) $\Sigma_j^+(M)$ has no sub-rep. isomorphic to S_j.

Proof. (a) Assume $M(r) = 0 \;\forall r \neq j$, then by the definition $M^+(r) = M(r) = 0 \;\forall r \neq j$, and

$$M^+(j) = Ker(M(\alpha_1), \dots, M(\alpha_n)) \text{ in } \prod_{i=1}^{t} M(i) = M(1) \times \dots \times M(t).$$

But each of these maps has the form $M(\alpha_i) : M(i) \to M(j)$ but $M(i) = 0 \;\forall i \neq j$, so $M(\alpha_i)$ is forced to be the zero-map and thus $Ker(M(\alpha_i)) = M(i) = 0$.
By the definition then $M^+(j) = 0 \subset M(1) \times \dots \times M(t)$.

Conversely, if $\Sigma_j^+(M) = 0$ then $\forall r \neq j$ we have
$$0 = M^+(r) = M(r)$$
directly from the definition of $\Sigma_j^+(M)$.

(b) Sup. $\Sigma_j^+(M)$ has a sub-rep. $\cong S_j$. Then we have

$$0 \neq (m_1, \dots, m_t) \in M^+(j), \text{ with } M^+(\bar{\alpha}_i)(m_1, \dots, m_t) = 0$$

for $i = 1, \dots, t$. But by definition, $M^+(\bar{\alpha}_i) : (m_1, \dots, m_t) \mapsto m_i$

and $(m_1, \dots, m_t) = 0$!

$\therefore \Sigma_j^+(M)$ has no sub-rep. $\cong S_j$. $\square$

The Reflection Σ_j^- at a Source.

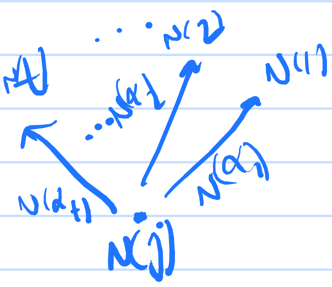
* Construction analogous to that of Σ_j^+ at a sink.

Def. Let j source in Q'. Take $\beta_1, \beta_2, \dots, \beta_t \in Q'_A$ with $s(\beta_K) = j$ for $K = 1, \dots, t$, s.t. $\beta_i : j \to i$. We write

$$\bar{\beta}_i : i \longrightarrow j$$

for $\bar{\beta}_i \in \sigma_j Q'_A$; $i = 1, \dots, t$.

Def. Let Q' quiver and assume j source of Q'. For rep. N of Q', define a rep. $\Sigma_j^-(N)$ of $\sigma_j Q'$ as follows. Set



$$N^-(r) = \begin{cases} N(r) & , r \neq j \\ (N(1) \times \cdots \times N(t)) / C_N & , r = j \end{cases}$$

where

$$C_N := \{ (N(\beta_1)(x), \dots, N(\beta_t)(x)) \mid x \in N(j) \} .$$
$$\subset N(1) \times \cdots \times N(t)$$

Define $N^-(\gamma) = N(\gamma)$ if $s(\gamma) \neq j$, and otherwise

$$\begin{aligned} N^-(\bar{\beta}_i) : N(i) &\longrightarrow N^-(j) \\ n_i &\longmapsto (0, \dots, 0, n_i, 0, \dots, 0) + C_N . \end{aligned}$$

$\Rightarrow \Sigma_j^-(N) = (\{N^-(i)\}_{i \in Q'_V}, \{N^-(\beta)\}_{\beta \in Q'_A})$ is a rep. for $\sigma_j Q'$.

Prop. Assume Q' quiver and j source of Q'. Let N be a rep. of Q'.

(a) $\Sigma_j^-(N) = 0$ iff $N(i) = 0 \ \forall i \neq j$.

(b) $\Sigma_j^-(N)$ has no direct summand $\cong S_j$.

Proof. Similar to the one for Σ_j^+ !.