Lecture 27 Root systems

Let E be a finite dimensional $\mathbb{R}$-vector space with a non-degenerate symmetric bilinear form

$$E \times E \xrightarrow{B} \mathbb{R}$$

$$(v, w) \longmapsto B(v, w) =: (v|w)$$

The reflection r_v attached to a non-zero $v \in E$ is

$$E \xrightarrow{r_v} E$$

$$w \longmapsto w - 2\frac{(v|w)}{(v|v)} v$$

A root system[1] is a finite subset $\Phi \subseteq E - \{0\}$ s.t.

$E = \langle \Phi \rangle_{\mathbb{R}}$ and for all $\alpha, \beta \in \Phi$

(1) $\alpha \in \mathbb{R}\beta \Rightarrow \alpha = \pm\beta$

(2) $\sigma_\alpha \Phi \subseteq \Phi$,

(3) $2\dfrac{(\alpha|\beta)}{(\alpha|\alpha)} =: \langle \alpha, \beta \rangle \in \mathbb{Z}$

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1 From the German *Wurzelsystem*. (Wilhelm Killing, 1889).

<u>Lemma</u> If θ is the angle between $\alpha, \beta \in \Phi$, then

$$\theta \in \left\{0, \pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}\right\} \qquad *$$

Proof

Note that

$$\mathbb{Z} \ni \underbrace{\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle}_{\substack{\| \\ 4}\ \Rightarrow\ \alpha = \pm\beta} = \frac{4(\alpha|\beta)}{|\alpha|^2 |\beta|^2} = 4\cos^2(\theta) \in [0, 4],$$

which gives $(*)$ $\square$

If a subset $\Delta = \{\alpha_1, \dots, \alpha_r\} \subseteq \Phi$ is a basis of E s.t.

$\forall \alpha \in \Phi$

$$\alpha = n_1 \alpha_1 + \dots + n_r \alpha_r$$

either $n_1, \dots, n_r \in \mathbb{Z}_{\geqslant 0}$ or $n_1, \dots, n_r \in \mathbb{Z}_{\leq 0}$, then

$$\Phi^+ := \{ \alpha = n_1 \alpha_1 + \dots + n_r \alpha_r \in \Phi \mid n_1, \dots, n_r \in \mathbb{Z}_{\geqslant 0} \}$$

is a set of *positive roots* and the elements of Δ are known as the corresponding *positive simple roots*.

<u>Lemma</u> Let $\alpha, \beta \in \Phi$. So

(i) if θ is strictly obtuse, then $\alpha + \beta \in \Phi$

(ii) if θ is strictly acute and $|\alpha| \leq |\beta|$, then $\alpha - \beta \in \Phi$

Proof (sketch)

Use the previous lemma $\square$

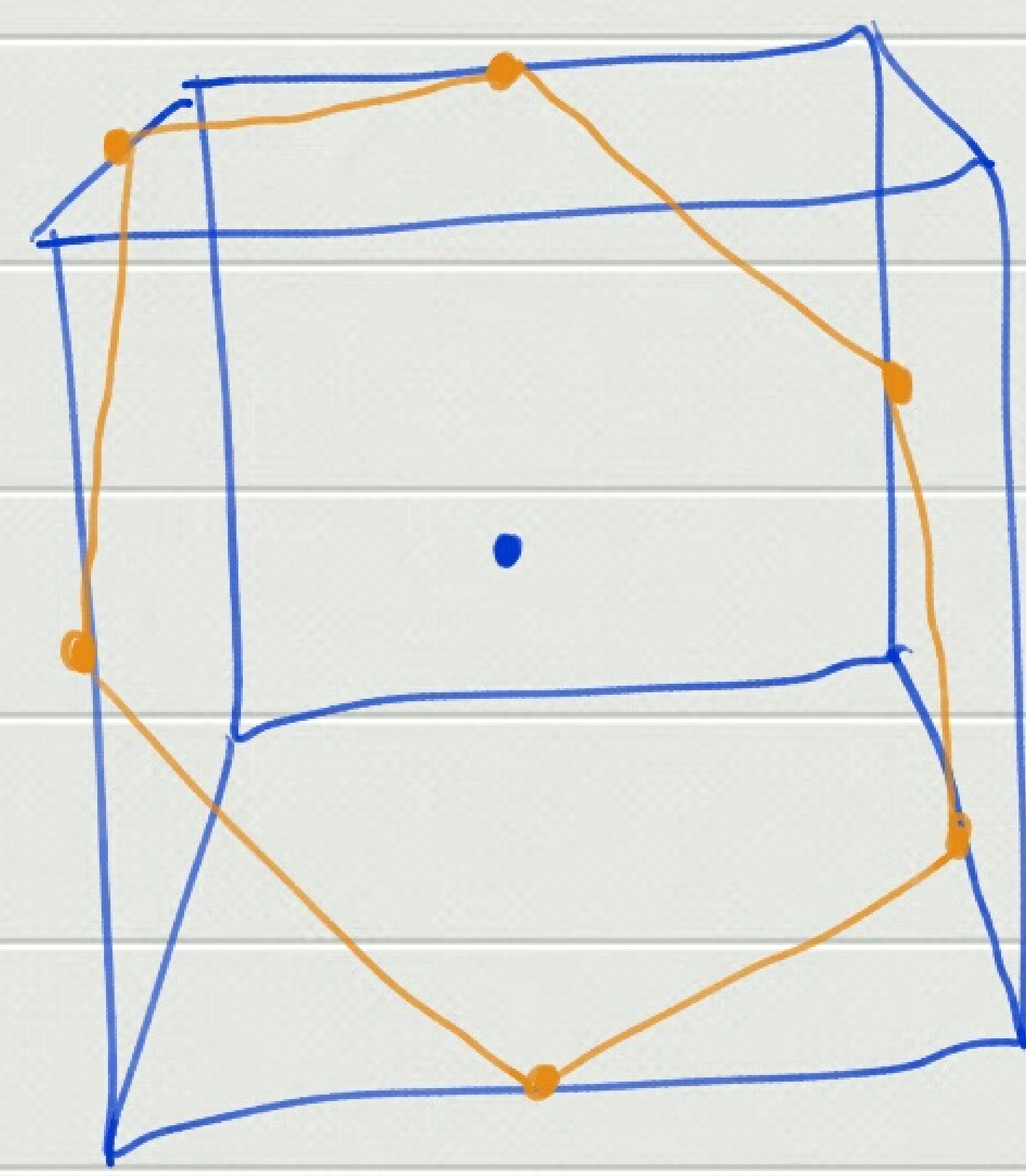
<u>Corollary</u> If $\alpha_i, \alpha_j \in \Delta$ then θ_{ij} is strictly obtuse.

Example Take

$$\Delta = \{\alpha_1, \alpha_2\} \subseteq E := \ker\left(\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{\varphi} & \mathbb{R} \\ (x_1, x_2, x_3) & \mapsto & x_1 + x_2 + x_3 \end{array}\right)$$

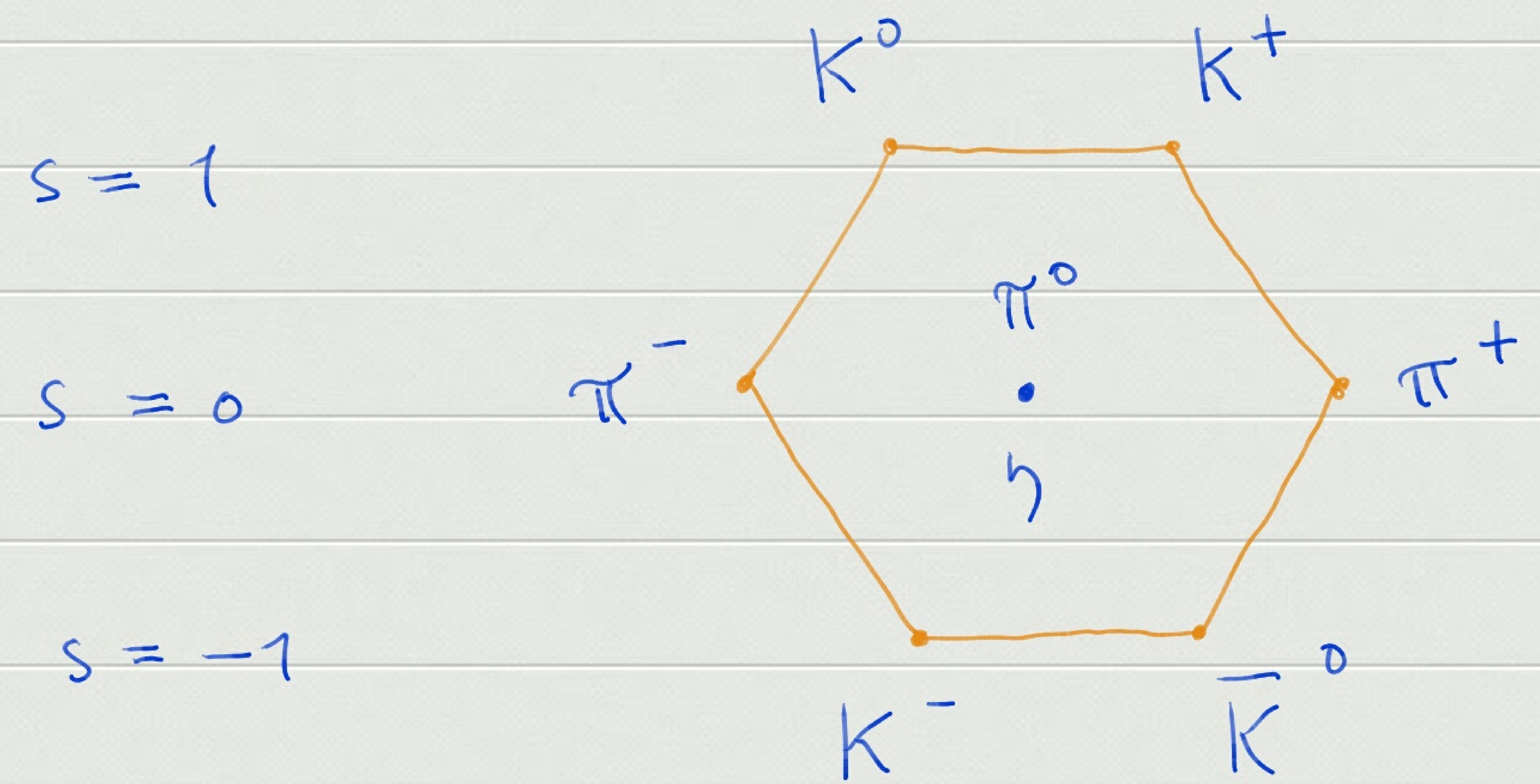
$\alpha_2 := (0, -1, 1)$

$(-1, 1, 0) =: \alpha_1$



We have $|\Phi| = 6$.

<u>Remark</u> The root system in the above example was used by Gell-Mann and Ne'eman (1961) in the *Eightfold way*, as in the *meson octet*



(strangeness)

More about this soon!

<u>Example</u> More generally, for each $r \in \{1, 2, 3, \dots\}$ take

$$\Delta = \{\alpha_1, \dots, \alpha_r\} \subseteq E := \ker\left(\begin{array}{ccc} \mathbb{R}^{r+1} & \xrightarrow{\varphi} & \mathbb{R} \\ (x_1, \dots, x_{r+1}) & \mapsto & \sum_{i=1}^{r+1} x_i \end{array}\right),$$

where

$$\alpha_1 := (-1, 1, 0, 0, \dots, 0, 0, 0),$$
$$\alpha_2 := (0, -1, 1, 0, \dots, 0, 0, 0),$$
$$\vdots$$
$$\alpha_r := (0, 0, 0, 0, \dots, 0, -1, 1).$$

We have $|\Phi| = n(n+1)$.

The Cartan[2] matrix attached to the root system Φ is

$$A_\Phi = \begin{bmatrix} \langle \alpha_1, \alpha_1 \rangle & \cdots & \langle \alpha_1, \alpha_r \rangle \\ \vdots & & \vdots \\ \langle \alpha_r, \alpha_1 \rangle & \cdots & \langle \alpha_r, \alpha_r \rangle \end{bmatrix} =: (a_{ij}) \in M_r(\mathbb{Z})$$

It's s.t.

$$\forall\, i,j \in \{1, \ldots, r\} : \left\{ \begin{array}{l} a_{ii} = 2 \\ a_{ij} \leq 0, \text{ if } i \neq j \\ a_{ij} = 0 \Rightarrow a_{ji} = 0 \end{array} \right\} \quad \star$$

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2 Élie Joseph Cartan.

Example The Cartan matrix for the root system Φ of the previous example is given by

$$A_{\Phi} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & & 0 & 0 \\ 0 & -1 & 2 & -1 & & & \vdots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & & & \\ & & & \ddots & \ddots & -1 & 0 & 0 \\ \vdots & \vdots & & & -1 & 2 & -1 & 0 \\ 0 & 0 & \cdots & & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & & 0 & 0 & -1 & 2 \end{pmatrix}$$