Lecture 28 Dynkin diagrams & root systems

Given a graph with no loops $\Gamma = (\Gamma_0, \Gamma_1)$ we'll construct a matrix $A \in M_r(\mathbb{Z})$ that satisfies $(\star)$. Write $\Gamma_0 = \{1, \ldots, r\}$, $d_{ij} := \left| \{ e \in \Gamma_1 \mid e \text{ connects } i \text{ and } j \} \right|$, and let $A_\Gamma = (a_{ij})$ with

$$a_{ij} := \begin{cases} -d_{ij}, & \text{if } i \neq j \\ 2, & \text{if } i = j \end{cases}$$

For example, for the Dynkin diagram A_4

we have

$$A_{\Gamma} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Given Γ as above, the set of roots of Γ is

$$\Phi_{\Gamma} := \{ v \in \mathbb{Z}^{r} \mid q_{\Gamma}(v) = 1 \},$$

where

$$q_{\Gamma}(v) = \frac{1}{2} v A_{\Gamma} v^{t}.$$

Let $\langle v, w \rangle_\Gamma$ be the symmetric bilinear form attached to q_Γ.

For each $j \in \Gamma_0$ we define

$$\sigma_j : \mathbb{Z} \longrightarrow \mathbb{Z}$$
$$v \longmapsto v - \langle v, e_j \rangle_\Gamma e_j$$

<u>Lemma</u> Then $\forall\, j \in \Gamma_0$

$$\sigma_j \Phi_\Gamma \subseteq \Phi_\Gamma .$$

Proof

We claim that σ_j preserves $\langle\ ,\ \rangle_\Gamma$. Indeed

$$\langle \sigma_j(v), \sigma_j(v)\rangle_\Gamma = \langle v - \langle v, e_j\rangle_\Gamma e_j, v - \langle v, e_j\rangle_\Gamma e_j\rangle_\Gamma =$$

$$\langle v, v\rangle_\Gamma - 2\underbrace{\langle v, e_j\rangle_\Gamma \langle e_j, v\rangle_\Gamma} + \langle v, e_j\rangle_\Gamma^2 \underbrace{\langle e_j, e_j\rangle_\Gamma}_{=2} = \langle v, v\rangle_\Gamma ,$$

Now note that

$$q_\Gamma(\sigma_j(v)) = \frac{1}{2}\langle \sigma_j(v), \sigma_j(v)\rangle = \frac{1}{2}\langle v, v\rangle_\Gamma = q_\Gamma(v).$$

Thus $q_\Gamma(v) = 1$ implies that $q_\Gamma(\sigma_j(v)) = 1$ $\square$

Conversely, we shall attach to each root system Φ a graph Γ_Φ. First we pick a set of simple roots

$$\Delta = \{\alpha_1, \dots, \alpha_r\}$$

(i) $\forall i \in \{1, \dots, r\}$ draw a node.

(ii) $\forall i,j \in \{1, \dots, r\}$ draw $\underbrace{\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle}_{\in \{0,1,2,3\}}$ edges.

We say that our root system Φ is reducible if $\Phi = \Phi_1 \dot{\cup} \Phi_2$ with $(\Phi_1 | \Phi_2) = 0$, and otherwise we say that Φ is irreducible.

<u>Thm</u> If the root system Φ is irreducible then Γ_Φ is a Dynkin diagram.