Lie Algebras

an introduction

Pedro Llerenas, Francisco Cruz

TABLE OF CONTENTS

Definition (Algebra).

An *algebra* over a field $\mathbb{F}$ is a vector space A over $\mathbb{F}$ together with a bilinear map,

$$A \times A \to A, \qquad (x, y) \mapsto xy.$$

Examples.

- $\mathrm{gl}(V)$ (the set of linear maps from V to V) with addition an composition.
- $\mathbb{H}$ the algebra of quaternions.
- $\mathbb{F}$ is a commutative algebra of dimension 1.

Definition (Lie Algebra).

Let $\mathbb{F}$ be a field. A *Lie algebra L* over $\mathbb{F}$ is an algebra whose bilinear operation *Lie bracket*

$$L \times L \to L, \qquad (x, y) \mapsto [x, y],$$

satisfies the following properties:

$$[x, x] = 0 \quad \text{for all } x \in L, \tag{L1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in L. \tag{L2}$$

Condition (L2) is known as the Jacobi identity. As the Lie bracket $[-, -]$ is bilinear, we have

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Hence condition $(L1)$ implies

$$[x, y] = -[y, x] \quad \text{for all } x, y \in L. \tag{L1'}$$

We also see that $(L1') \implies (L1)$ if $char(\mathbb{F}) \neq 2$.

Examples.

- $\mathbb{R}^3$ with the cross product $(x, y) \mapsto x \wedge y$ forms a Lie algebra denoted by $\mathbb{R}^3_\wedge$.
- Any vector space V has a Lie bracket defined $[x, y] = 0$ for all $x, y \in V$. This is the *abelian* Lie algebra structure on V. In particular $\mathbb{F}$ is a 1-dimensional Lie algebra.
- If we define $[x, y] := x \circ y - y \circ x$ in gl(V), then this is the Lie algebra called the *general linear algebra*.
- In general, if we have an associative algebra A over $\mathbb{F}$, then we may define a new bilinear opereation $[x, y] = xy - yx$. A together with $[-, -]$ is a Lie algebra.

Definition (Homomorphism).

If L_1 and L_2 are Lie algebras over $\mathbb{F}$, we say that a linear map $\varphi : L_1 \to L_2$ is a *homomorphism* if

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \quad \text{for all } x, y \in L_1.$$

Notice that in the above equation the first Lie bracket is taken in L_1 and the second Lie bracket is taken in L_2. We say that φ is a *isomorphism* if it is a bijective homomorphism.

Example.

The *adjoint homomorphism* $\operatorname{ad} : L \to \mathrm{gl}(L)$ is defined by

$$(\operatorname{ad} x)(y) := [x, y] \quad \forall\, y \in L.$$

The linearity of ad and $(\operatorname{ad} x) \in \mathrm{gl}(L)$ both follow from the bilinearity of the Lie bracket, and

$$\operatorname{ad}([x, y]) = \operatorname{ad} x \circ \operatorname{ad} y - \operatorname{ad} y \circ \operatorname{ad} x$$

is equivalent to the Jacobi identity.

Definition (Lie subalgebra).

A *Lie subalgebra* of L is defined to be a vector subspace $K \subseteq L$ such that

$$[x, y] \in K \quad \text{for all } x, y \in K.$$

Examples.

- Let $\mathrm{sl}(n, \mathbb{F})$ be the subspace of $\mathrm{gl}(n, \mathbb{F})$ consisting of all matrices of trace 0. For arbitrary square matrices x and y, the matrix $xy - yx$ has trace 0, so $[x, y] = xy - yx$ defines a Lie algebra structure on $\mathrm{sl}(n, \mathbb{F})$.
- Let $\mathrm{b}(n, \mathbb{F})$ be the upper triangular matrices in $\mathrm{gl}(n, \mathbb{F})$. This is a Lie algebra with the same Lie bracket as $\mathrm{gl}(n, \mathbb{F})$.
- If $\varphi : L_1 \to L_2$ is a homomorphism, then $\operatorname{im} \varphi$ is a Lie subalgebra of L_2.

Definition (Derivation).

Let A be an algebra over $\mathbb{F}$, a *derivation* of A is a linear map $D : A \to A$ such that

$$D(ab) = aD(b) + D(a)b \quad \forall\, a, b \in A.$$

Example.

- The map $(\operatorname{ad} x) : L \to L$ is a derivation of L since by the Jacobi identity

$$(\operatorname{ad} x)[y, z] = [x, [y, z]] = [y, [x, z]] + [[x, y], z] = [y, (\operatorname{ad} x)z] + [(\operatorname{ad} x)y, z].$$

Der A, the set of derivations of A is closed under addition and scalar multiplication, and contains the zero map. Hence Der A is a vector subspace of $\mathfrak{gl}(A)$.

Moreover it can be proven that if D and E are derivations $[D, E] = D \circ E - E \circ D$ is also a derivation, thus Der A is a Lie subalgebra of $\mathfrak{gl}(A)$.

Definition (Ideal).

An *ideal* of a Lie algebra L is a subspace I of L such that

$$[x, y] \in I \quad \text{for all } x \in L,\ y \in I.$$

Condition (L1’) implies we do not need to distinguish between left ideal and right ideals.

Examples.

- The Lie algebra L is itself an ideal of L. At the other extreme, $\{0\}$ is an ideal of L. These are the *trivial ideals* of L.
- If $\varphi : L_1 \to L_2$ is a homomorphism, then $\ker \varphi$ is an ideal of L_1.
- $\mathrm{sl}(n, \mathbb{F})$ is an ideal of $\mathrm{gl}(n, \mathbb{F})$.
- Let $Z(L) := \{x \in L : [x, y] = 0\ \forall y \in L\}$. This is called the *centre* of L and it is an ideal since $\ker \mathrm{ad} = Z(L)$. Moreover $Z(L) = L$ if and only if L is abelian.

It is clear that every ideal is a subalgebra, but not every subalgebra is an ideal, as seen in the following counterexample:

Counterexample.

- b$(2, \mathbb{F})$ is a subalgebra of gl$(2, \mathbb{F})$, but note that

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$[e_{21}, e_{11}] = e_{21} \notin b(2, \mathbb{F}).$$

Suppose that I and J are ideals of a Lie algebra L. There are several ways we can construct new ideals from I and J.

Examples.

- $I \cap J$ is a subspace of L, and $[x, y] \in I \cap J$ for $x \in L$ and $y \in I \cap J$, since they are each individually an ideal, then $I \cap J$ is an ideal of L.
- $I + J$ is a subspace of L. Let $v \in L$ and $u = i + j \in I + J$, then

$$[v, u] = [v, i + j] = [v, i] + [v, j] \in I + J.$$

  Therefore $I + J$ is an ideal of L.

- The product of ideals defined by $[I, J] = \text{span}\{[x, y] : x \in I,\ y \in J\}$ is by definition a subspace. To show it is an ideal of L, let $x \in I, y \in J$ and $u \in L$, then the Jacobi identity gives

$$[u, [x, y]] + [x, [y, u]] + [y, [u, x]] = 0 \iff [u, [x, y]] = [x, [u, y]] + [[u, x], y]$$

Where $[u, y] \in J$ and $[u, x] \in I$, hence $[x, [u, y]], [[u, x], y] \in [I, J]$ and their sum belongs to $[I, J]$.
Let $t \in [I, J]$, then, for $x_i \in I$ and $y_i \in J$, we may write

$$t = \sum c_i [x_i, y_i]$$

where the c_i are scalars. Now, for any $u \in L$, we have

$$[u, t] = \left[u, \sum c_i [x_i, y_i]\right] = \sum c_i \, [u, [x_i, y_i]]$$

hence $[u, t] \in [I, J]$, so $[I, J]$ is an ideal of L.

In the construction of $[I, J]$, the particular case $I = J = L$ is denoted as $L' = [L, L]$. This is of course an ideal, but is usually referred to as the *derived algebra* of L.

Examples.

- $\mathrm{sl}(2, \mathbb{C})' = \mathrm{sl}(2, \mathbb{C})$
- If L is abelian, then $L' = 0$.

If I is an ideal of the Lie algebra L, then I is a vector subspace of L, and so we may consider the quotient vector space $L/I = \{z + I : z \in L\}$.

A Lie bracket on L/I can be defined by

$$[w + I, z + I] := [w, z] + I \quad w, z \in L.$$

To be sure that the Lie bracket is well defined we must check that $[w, z] + I$ does not depend on the particular coset representatives w and z.

Suppose $w + I = w' + I$ and $z + I = z' + I$. Then $w' - w \in I$ and $z' - z \in I$. By bilinearity of the Lie bracket in L,

$$\begin{aligned}
[w', z'] &= [w + w' - w, z'] \\
&= [w, z'] + [w' - w, z'] \\
&= [w, z + z' - z] + [w' - w, z'] \\
&= [w, z] + [w, z' - z] + [w' - w, z'].
\end{aligned}$$

Thus $[w' + I, z' + I] = [w, z] + I$.

Definition (Quotient Algebra).

If I is an ideal of a Lie Algebra L, then L/I is a Lie algebra with Lie bracket

$$[w+I, z+I] = [w,z] + I.$$

Theorem (Isomorphism theorems)

(a) *Let $\varphi : L_1 \to L_2$ be a homomorphism of Lie algebras, Then* $\ker\varphi$ *is an ideal of L_1 and* $\operatorname{im}\varphi$ *is a subalgebra of L_2, and*

$$L_1/\ker\varphi \cong \operatorname{im}\varphi.$$

(b) *If I and J are ideals of a Lie algebra, then*

$$(I+J)/J \cong I/(I\cap J).$$

(c) *Suppose that I and J are ideals of a Lie algebra L such that $I \subset J$. Then J/I is an ideal of L/I and*

$$(L/I)/(J/I) \cong L/J.$$

Low-Dimensional Lie Algebras

We begin this section to identify how many non-isomorphic Lie algebras there are and what approaches can be used to classify them. We will look at Lie algebras of dimensions 1, 2 and 3.

Abelian Lie algebras are easily understood, as two abelian Lie algebras of the same dimension over the same field are isomorphic, henceforth we consider non-abelian Lie algebras.

If L is a non-abelian Lie algebra, then its derived algebra L' is non-zero and its centre $Z(L)$ is a proper ideal.

It is clear that any 1-dimensional Lie algebra is abelian since $[\alpha x, \beta x] = \alpha\beta[x, x] = 0$.

For dimension 2, we have the following theorem.

Theorem (Classification of two-dimensional Lie algebras)

Let $\mathbb{F}$ be any field. Up to isomorphism there is a unique two-dimensional non-abelian Lie algebra over $\mathbb{F}$. This Lie algebra has a basis $\{x, y\}$ such that its Lie bracket is described by $[x, y] = x$. The centre of this Lie algebra is 0.

Proof

Suppose L is a non-abelian Lie algebra of dimension 2 over $\mathbb{F}$, then L' cannot be more than 1-dimensional, since if $\{x, y\}$ is a basis of L, then L' is spanned by $[x, y]$. We conclude it is 1-dimensional, as it is non-zero, otherwise L would be abelian.

Take a non-zero element $x \in L'$ and extend it to a vector space basis $\{x, \bar{y}\}$ of L, then we have a non-zero element $[x, \bar{y}] \in L'$, otherwise L would be abelian, thus we may write $[x, \bar{y}] = \alpha x$. If we replace $y := \alpha^{-1}\bar{y}$, the structure of L is preserved and we obtain

$$[x, y] = x.$$

It remains to verify that this bracket satisfies the Jacobi identity. Let $x, y, z \in L$, where $z = ax + by$

$$\begin{aligned}
[x,[y,ax+by]]+&[y,[ax+by,x]]+[ax+by,[x,y]] \\
&= [x,[y,ax]+[y,by]]+[y,[ax,x]+[by,x]]+[ax,[x,y]]+[by,[x,y]] \\
&= [x,[y,ax]]+[y,[by,x]]+[ax,x]+[by,x] \\
&= [x,-a[x,y]]+[y,-b[x,y]]-b[x,y] \\
&= -a[x,x]-b[y,x]-b[x,y] \\
&= 0.
\end{aligned}$$

□

If L is a non-abelian 3-dimensional Lie algebra over a field $\mathbb{F}$, then we know only that the derived algebra L' is non-zero. It might be of dimensions 1,2 or 3.

We will only consider when it has dimension 1. We will try to relate $Z(L)$ to L' to obtain further information on the classification of such Lie algebra.

Theorem (The Heisenberg Algebra)

There is a unique 3-dimensional Lie algebra L such that L' is 1-dimensional and $L' \subset Z(L)$.

Moreover it has a basis f, g, z, where $[f, g] = z \in Z(L)$.

Proof

Take any $f, g \in L$ such that $[f, g]$ is non-zero; since L' is 1-dimensional, the commutator $[f, g]$ spans L' and because $L' \subset Z(L)$, $[f, g]$ commutes with all elements of L.
Set $z := [f, g]$, then f, g, z form a basis of L and by bilinearity, all other Lie brackets are already fixed. □

It only remains to investigate the case where L' is 1-dimensional and $L' \not\subset Z(L)$. To do this we'll need the following result

Lemma

$Z(L_1 \oplus L_2) = Z(L_1) \oplus Z(L_2)$ and $(L_1 \oplus L_2)' = L_1' \oplus L_2'$, where

$$[(x_1, x_2), (y_1, y_2)] := ([x_1, y_1], [x_2, y_2]).$$

Proof

Note that

$$\begin{aligned}
(x_1, x_2) \in Z(L_1 \oplus L_2) &\iff [(x_1, x_2), (y_1, y_2)] = 0 \; \forall \, y \in L \\
&\iff ([x_1, y_1], [x_2, y_2]) = 0 \; \forall \, y_1 \in L_1, \; y_2 \in L_2 \\
&\iff x_1 \in Z(L_1) \text{ and } x_2 \in Z(L_2) \\
&\iff (x_1, x_2) \in Z(L_1) \oplus Z(L_2)
\end{aligned}$$

$$
\begin{aligned}
(x_1, x_2) \in (L_1 \oplus L_2)' &\iff (x_1, x_2) \in span\{[a, b] : a, b \in L_1 \oplus L_2\} \\
&\iff (x_1, x_2) = \sum \alpha_k [a_k, b_k] \\
&\iff (x_1, x_2) = \sum \alpha_k [(a_{k,1}, a_{k,2}), (b_{k,1}, b_{k,2})] \\
&\iff (x_1, x_2) = \sum \alpha_k ([a_{k,1}, b_{k,1}], [a_{k,2}, b_{k,2}]) \\
&\iff (x_1, x_2) = \left(\sum \alpha_k [a_{k,1}, b_{k,1}], \sum \alpha_k [a_{k,2}, b_{k,2}]\right) \\
&\iff x_1 \in L_1' \text{ and } x_2 \in L_2' \\
&\iff (x_1, x_2) \in L_1' \oplus L_2'
\end{aligned}
$$

$\square$

We'll first construct a Lie algebra with the desired properties by using the above lemma.

Consider $L = L_1 \oplus L_2$, where L_1 is 2-dimensional and non-abelian (the Lie algebra descirbed by $[x, y] = x$) and L_2 is 1-dimensional. By the lemma

$$L' = L_1' \oplus L_2' = L_1'$$

hence L' is 1-dimensional. Moreover, $Z(L) = Z(L_1) \oplus Z(L_2) = L_2$, therefore L' is not contained in $Z(L)$.

Theorem

Let $\mathbb{F}$ be a field. There is a unique 3-dimensional Lie algebra over $\mathbb{F}$ such that L' is 1-dimensional and L' is not contained in $Z(L)$. This Lie algebra is the direct sum of the 2-dimensional non-abelian Lie algebra with the 1-dimensional Lie algebra.

Proof

Pick a non-zero element $x \in L' \not\subset Z(L)$, thus there must exist $y \in L$ such that $[x, y] \neq 0$, then they are linearly independent. By the Theorem of Classification of two-dimensional Lie algebras, we may assume that $[x, y] = x$. We then extend $\{x, y\}$ to a basis $\{x, y, w\}$ of L. Since x spans L', there exists scalars α, β such that

$$[x, w] = \alpha x, \qquad [y, w] = \beta x.$$

We claim that L contains a non-zero central element z which is not in the span of x and y. For $z = \lambda x + \mu y + \nu w \in L$,

$$[x, z] = [x, \lambda x + \mu y + \nu w] = \mu x + \nu \alpha x,$$
$$[y, z] = [y, \lambda x + \mu y + \nu w] = -\lambda x + \nu \beta x.$$

Taking $\lambda = \beta$, $\mu = -\alpha$ and $\nu = 1$ we have $[x, z] = [y, z] = 0$ and z is not in the space spanned by x and y. Hence L =span$\{x, y\}\oplus$span$\{z\}$ as desired. $\square$

We'll end by studing some properties of the important Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (the 2×2 matrices with complex entries and zero trace).

Theorem

$$\mathfrak{gl}(2, \mathbb{C}) / \mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}$$

Proof

Notice that $\operatorname{tr} : \mathfrak{gl}(2, \mathbb{C}) \to \mathbb{C}$ is a Lie algebra homomorphism, for if $x, y \in \mathfrak{gl}(2, \mathbb{C})$, then

$$\operatorname{tr}[x, y] = \operatorname{tr}(xy - yx) = \operatorname{tr} xy - \operatorname{tr} yx = 0$$

so $\operatorname{tr}[x, y] = [\operatorname{tr} x, \operatorname{tr} y] = 0$.
Clearly $\ker \operatorname{tr} = \mathfrak{sl}(2, \mathbb{C})$. By the first isomorphism theorem we have

$$\mathfrak{gl}(2, \mathbb{C}) / \mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}.$$

$\square$

Theorem

The following matrices form a basis of $\mathfrak{sl}(2, \mathbb{C})$.

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof

Notice that

$$\alpha e + \beta f + \gamma h = \begin{pmatrix} \gamma & \alpha \\ \beta & -\gamma \end{pmatrix}$$

so the list spans $\mathfrak{sl}(2, \mathbb{C})$, and since it is linearly independent, it is a basis. □

Moreover, we have

$$[e, f] = h, \quad [h, f] = -2f, \quad [h, e] = 2e.$$

Theorem

$\mathfrak{sl}(2, \mathbb{C})$ *has no non-trivial ideals.*

Proof

First note that because $[h, f] = -2f$ and $[h, e] = 2e$, it suffices to show that if $I \neq 0$, then $h \in I$.

Suppose $I \neq 0$, and let $x = \alpha e + \beta f + \gamma h$ be a non-zero element of I. Now consider

$$(\operatorname{ad} h)(x) = [h, \alpha e + \beta f + \gamma h] = 2\alpha e - 2\beta f$$
$$(\operatorname{ad} h)^2(x) = [h, 2\alpha e - 2\beta f] = 4\alpha e + 4\beta f$$

Since I and ideal, $(\operatorname{ad} h)^2(x) \in I$, thus $\gamma h = x - \frac{1}{4}(\operatorname{ad} h)^2(x)$ is also in I. Hence $\gamma = 0$ or $h \in I$.

If $\gamma = 0$, then $(\operatorname{ad} e)(x) = \beta h$, again $\beta = 0$ or $h \in I$. If $\gamma = \beta = 0$, then $(\operatorname{ad} f)(x) = -\alpha h$. Since x is non-zero, we are done. □

Corollary

$$Z(\mathsf{sl}(2\mathbb{C})) = 0, \qquad \mathsf{sl}(2, \mathbb{C})' = \mathsf{sl}(2, \mathbb{C}).$$

Proof

Both $Z(\mathsf{sl}(2\mathbb{C}))$ and $\mathsf{sl}(2, \mathbb{C})'$ are ideals of $\mathsf{sl}(2, \mathbb{C})$, since $\mathsf{sl}(2, \mathbb{C})$ is not abelian $Z(\mathsf{sl}(2\mathbb{C})) \neq \mathsf{sl}(2, \mathbb{C})$, thus $Z(\mathsf{sl}(2\mathbb{C})) = 0$ and because $\mathsf{sl}(2, \mathbb{C})' \neq 0$, $\mathsf{sl}(2, \mathbb{C})' = \mathsf{sl}(2, \mathbb{C})$. □

It also can be shown that $\mathsf{sl}(2, \mathbb{C})$ is the unique 3-dimensional Lie algebra over $\mathbb{C}$ such that $L = L'$.