Lecture 31. Basics of representation theory

For each V be a k-vector space we have the Lie algebra with

$$\mathfrak{gl}(V) := (\mathrm{End}(V), [\cdot;\cdot])$$

$[f, g] := f \circ g - g \circ f$, $\forall$ $f, g \in \mathrm{End}(V)$. If $\dim(V) = n < \infty$ and $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ is a basis for V, then we have a ring isomorphism $\mathrm{End}(V) \cong M_n(k)$ and thus a Lie algebra isomorphism $\mathfrak{gl}(V) \cong \mathfrak{gl}_n(k)$, where

$$\mathfrak{gl}_n(k) := \left(M_n(k), [\cdot,\cdot] \right)$$

with $[A,B] := AB - BA$. A *representation* of a given Lie algebra L on a vector space V is a morphism

$$\rho : L \longrightarrow \mathfrak{gl}(V)$$

<u>Example</u> For each Lie algebra L we have the adjoint rep^n

$$L \xrightarrow{\mathrm{ad}} \mathfrak{gl}(L)$$

$$x \longmapsto \begin{pmatrix} L \longrightarrow L \\ y \longmapsto [x,y] \end{pmatrix}$$

Recall that a derivation over k on any algebra $\mathcal{A}$, with $B: \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, is a k-linear map

$$\delta : \mathcal{A} \longrightarrow \mathcal{A}$$

That satisfies Liebniz rule

$$\delta B(x,y) = B(\delta x, y) + B(x, \delta y),$$

$\forall\, x, y \in \mathcal{A}$. The set $\mathrm{Der}(\mathcal{A})$ of all such derivations is subspace $\mathrm{Der}(\mathcal{A}) \leqslant \mathrm{End}(\mathcal{A})$. For a Lie algebra L we have $\mathrm{ad}(L) \leqslant \mathrm{Der}(L)$.

Given a Lie algebra L, an L-module is a vector space V together with a k-bilinear map

$$L \times V \longrightarrow V$$
$$(x, v) \longmapsto x \cdot v$$

s.t. $\forall\ x, y \in L,\ v \in V$

$$[x, y] \cdot v := x \cdot (y \cdot v) - y \cdot (x \cdot v).$$

Any L-module V yields a rep'n of L on V

$$\begin{aligned} L &\xrightarrow{\rho} \mathfrak{gl}(V) \\ x &\longmapsto \begin{pmatrix} V \longrightarrow V \\ y \longmapsto x \cdot y \end{pmatrix} \end{aligned}$$

Conversely, given a rep'n ρ of a Lie algebra L on a vector space V, we have a L-module V,

$$\begin{aligned} L \times V &\longrightarrow V \\ (x, y) &\longmapsto x \cdot y := \rho_x(y) \end{aligned}$$

Given L-modules V, W, their tensor product $V \otimes_k W$ is again an L-module by letting $\forall\, x \in L$, $v \in V$, $w \in W$

$$x \cdot (v \otimes w) := (x \cdot v) \otimes w + v \otimes (x \cdot w)$$

and extending to all $V \otimes_k W$ by k-linearity. Also the dual space $V^* := \mathrm{Hom}_k(V, k)$ is an L-module in a natural way: $\forall\, x \in L$, $\varphi \in V^*$, $v \in V$,

$$(x \cdot \varphi)(v) := -\varphi(x \cdot v).$$

In particular, we have an L-module isomorphism

$$\begin{aligned} V^* \otimes W &\xrightarrow{\sim} \mathrm{Hom}_k(V, W) \\ \varphi \otimes w &\longmapsto \left(\begin{aligned} V &\longrightarrow W \\ v &\longmapsto \varphi(v) w \end{aligned} \right) \end{aligned}$$

where the right-hand side has structure of L-module given by

$$(x \cdot \psi)(v) := x \cdot (\psi(v)) - \psi(x \cdot v),$$

$\forall\, x \in L$, $\psi \in \mathrm{Hom}_k(V, W)$, $v \in V$.

We say that an L-module V is *simple* if it has no nontrivial L-submodules $V' \leq V$. We say that L itself is *simple* its adjoint rep^n is simple and $\dim(L) > 1$.

We say that V is *semisimple* $V = \bigoplus_i V_i$ where each V_i is simple. The Lie algebra is *reductive* if ad is semisimple.

Fix a Lie algebra L and consider the category $\mathcal{E}_L$ whose objects are Lie algebra homomorphisms

$$\begin{array}{c} A \\ f \uparrow \\ L \end{array},$$

where A is an associative (unital) k-algebra with Lie bracket

$$[a,b] := ab - ba,$$

$\forall\, a,b \in A$ and for each pair of objects

$$\begin{tikzcd} A_1 & & A_2 \\ & L \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd} \in \mathrm{ob}(\mathcal{C}_L)$$

a morphism is a k-algebra homomorphism $A_1 \xrightarrow{g} A_2$ s.t.

$$\begin{tikzcd} A_1 \arrow[rr, "g"] & & A_2 \\ & L \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

commutes. Given any category D, a universally repelling object $U \in \mathrm{ob}(D)$ is s.t. $\forall A \in \mathrm{ob}(D) : |\mathrm{Hom}(U, A)| = 1$.

The universal enveloping algebra $\mathcal{U}(L)$ of L is a universally repelling object of $\mathcal{C}_L$, i.e. $\forall \; \begin{matrix} A \\ \uparrow \\ L \end{matrix} \in ob(\mathcal{C}_L) \; \exists !$ arrow

$\hat{f} : \mathcal{U}(L) \longrightarrow A$ of Rng_k that makes the diagram

$$\begin{tikzcd} \mathcal{U}(L) \arrow[r, "\hat{f}"] & A \\ L \arrow[u, "\iota"] \arrow[ur, "f"'] & \end{tikzcd}$$

commute, i.e. $\mathrm{Hom}_{\mathrm{Lie}}(L, A) \cong \mathrm{Hom}_{\mathrm{Rng}_k}(\mathcal{U}(L), A)$.

$$f \longmapsto \hat{f}$$

In particular, the above yields an equivalence of categories

$$\{ L\text{-moduls} \} \longleftrightarrow \{ \mathcal{U}(L)\text{-modules} \}$$

We'll show that $\mathcal{U}(L)$ exists by actually constructing it. Let

$$\mathcal{U}(L) := T(L) / I,$$

where $T(L) := \bigoplus_{n \geq 0} L^{\otimes n} = k \oplus L \oplus (L \otimes L) \oplus (L \otimes L \otimes L) \oplus \dots$

and $I = \langle x \otimes y - y \otimes x - [x,y] \mid x, y \in \rangle$.

<u>Thm</u> (Poincaré–Birkhoff–Witt) Let $\mathcal{B} = \{\beta_i\}_{i \in I}$ be a basis for L and pick a total order for I. Then

$$\{ \iota(\beta_{i_1})\iota(\beta_{i_2})\cdots\iota(\beta_{i_k}) \mid i_1 \leq i_2 \leq \cdots \leq i_k \}$$

is a basis for $\mathcal{U}(L)$. Moreover, $\iota : L \to \mathcal{U}(L)$ is injective.