Lecture 32 Some irred'l rep'ns

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field k s.t. $\chi(k) = 0$, e.g. $k = \mathbb{C}$.

A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$ s.t.

$$\mathfrak{h} = \{ x \in \mathfrak{g} \mid \mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g} \text{ is diagonalisable} \}$$

<u>Example</u> Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_n \end{pmatrix} \;\middle|\; \sum_{i=1}^{n} a_i = 0 \right\}$$

We have the direct sum decomposition

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda$$

where

$$\mathfrak{g}_\lambda = \{ x \in \mathfrak{g} \mid \forall h \in \mathfrak{h} : \mathrm{ad}_h(x) = \lambda(h) x \}$$

<u>Thm</u> The set

$$\Phi := \{ \lambda \in \mathfrak{h}^* - \{0\} \mid \mathfrak{g}_\lambda \neq 0 \}$$

is a root system. Moreover, $\forall \lambda \in \Phi$ we have $\dim(\mathfrak{g}_\lambda) = 1$.

Consider a representation of $\mathfrak{g}$ on V, i.e. a Lie algebra homomorphism

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}_{\mathbb{C}}(V)$$

and let for each $\lambda \in \mathfrak{h}^*$

$$V_\lambda = \{ v \in V \mid \forall h \in \mathfrak{h} : \rho_h(v) = \lambda(h) v \}.$$

<u>Thm</u> There is a bijection

$$\{ \lambda \in \mathfrak{h}^* \text{ dominant} \} \xrightarrow{\sim} \{ \text{eq. cl. of irred'l rep's of } \mathfrak{g} \}$$

$$\lambda \longmapsto [V_\lambda]$$

Let $\Lambda = \langle \Phi \rangle_{\mathbb{Z}}$, where Φ is the root system attached to $\mathfrak{g}$.
We say that the element $\lambda \in \Lambda$ is dominant if $\forall\ \gamma \in \Delta^+$
$(\lambda | \gamma) \geqslant 0$.

<u>Example</u> For $\mathfrak{sl}_3(\mathbb{C})$ we have $\Phi = A_2$ and the dominant weights are shown in green: