Lecture 1 Affine varieties

Let K denote a perfect field, i.e. either the characteristic $\chi(K)$ of K is 0 or $\chi(K) = p > 0$ and

$$\begin{aligned} K &\xrightarrow{\varphi} K \\ a &\longmapsto a^p \end{aligned} \qquad \text{(Frobenius homomorphism)}$$

is an automorphism.

Example $K = \mathbb{Q}$

Example $K = \mathbb{F}_{p^s}$, the finite field s.t. $|K| = p^s$.

We shall denote $\overline{K}$ a fixed algebraic closure of K and

$$G_K := \{ \sigma \in \mathrm{Aut}(\overline{K}) \mid \sigma|K = id \}.$$

This is the Galois group of $\overline{K}$ over K.

<u>Lemma</u> If $x \in \overline{K}$ is s.t. $\sigma(x) = x$, for all $\sigma \in G_K$, then $x \in K$.

<u>Example</u> Let $K = \mathbb{R}$, so $\overline{K} = \mathbb{C}$ and $G_K = \{1, \kappa\}$,

where κ is complex conjugation

$$a+bi \longmapsto a-bi,$$

where $a, b \in \mathbb{R}$. Moreover, we have $\forall\, z \in \mathbb{C}$

iff $\kappa(z) = z$.

<u>Remark</u> The field ext'n $\mathbb{C}/\mathbb{R}$ has degree $[\mathbb{C} : \mathbb{R}] = 2$, as $\mathcal{B} = \{1, i\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$. But in in general $[\overline{K} : K]$ is not finite. For example, if $K = \mathbb{F}_p$, where p is prime, then

$$[\mathbb{F}_{p^s} : \mathbb{F}_p] = s,$$

$\forall\, s \in \{1, 2, 3, \dots\}$. Therefore $[\overline{K} : K]$ is not finite.

Fix a non-negative integer n.

__Defn__ The affine n-space is

$$\mathbb{A}^n := \mathbb{A}^n(\bar{K}) = \{ P = (x_1, \dots, x_n) : x_i \in \bar{K} \} ;$$

its set of K-rational points is

$$\mathbb{A}^n(K) = \{ P = (x_1, \dots, x_n) : x_i \in K \}.$$

For each ideal $I \leq \bar{K}[X_1, \dots, X_n]$ we define the algebraic set

$$V := V(I) := \{ p \in \mathbb{A}^n \mid \forall f \in I : f(p) = 0 \}.$$

On the other hand, for each $V \subseteq \mathbb{A}^n$ the ideal of V is

$$\mathcal{I}(V) := \{ f \in \bar{K}[X_1, \dots, X_n] \text{ s.t. } \forall p \in V : f(p) = 0 \},$$

which is clearly a radical ideal, i.e.

$$\operatorname{rad}(\mathcal{I}(V)) := \{ f \in \bar{K}[X_1, \dots, X_n] \mid \exists n \geq 1 : f^n \in \mathcal{I}(V) \} = \mathcal{I}(V).$$

Let $\mathcal{V}_n$ denote the set of all algebraic subsets $V \subseteq \mathbb{A}^n$ and let $\mathcal{I}_{rad,n}$ denote the set of all radical ideals $I \subseteq \overline{K}[X_1, \ldots, X_n]$, i.e. $\mathrm{rad}(I) = I$. We have an inclusion reversing map

$$\begin{aligned} \mathcal{V}_n &\xrightarrow{\mathcal{I}} \mathcal{I}_{rad,n} \\ V &\longmapsto \mathcal{I}(V) \end{aligned}$$

s.t. $\mathcal{V} \circ \mathcal{I} = 1_{\mathcal{V}_n}$. This means that $\mathcal{I}$ is injective.

Hilbert's Nullstellensatz says that $\mathcal{I}$ is also surjective.

We say that an algebraic set $V \subseteq \mathbb{A}^n$ is a *variety* if the quotient ring

$$\overline{K}[V] := \overline{K}[X_1, \dots X_n] / I(V)$$

is an integral domain. We have a ring embedding

$$\overline{K}[V] \hookrightarrow \mathrm{Maps}(V, \overline{K})$$

$$\overline{f} = f(X_1, \dots, X_n) + I(V) \longmapsto \left(\begin{array}{c} V \xrightarrow{\varphi_f} \overline{K} \\ P \longmapsto f(P) \end{array} \right)$$

as $\varphi_f = \varphi_g \iff f - g \in I(V)$. In other words, the ring $\bar{K}[V]$ is canonically isomorphic to the subring of $\mathrm{Maps}(V, \bar{K})$ of polynomial functions.

<u>Remark</u> The above discussion allows us to regard the elements of $\bar{K}[V]$ as functions. But the fact that its elements are cosets $\bar{f}$, with $f \in \bar{K}[X_1, \dots, X_n]$ allows the Galois group G_K to act on $\bar{K}[V]$, a fact that is of paramount importance.

We shall find interesting to look at the larger field of fractions $\bar{K}(V)$ of $\bar{K}[V]$, in a similar spirit as in the study of meromorphic functions. In order to deal at once with all the possible maximal domains of definition of

$$P \mapsto h(P),$$

where $h = \frac{f}{g} \in \bar{K}(V)$ with $f, g \in \bar{K}[V]$ s.t. $g \neq 0$,

we consider the topology on V with basis

$$D_g := V \setminus V_g,$$

where V_g denotes the set of zeros of g on V.

Lemma The topology that we have just defined is the one where a $\mathcal{U} \subseteq V$ is open iff

$$\mathcal{U} = \mathcal{U}' \cap V,$$

where $\mathcal{U}' \subseteq \mathbb{A}^n$ is the complement of some algebraic subset of $\mathbb{A}^n$.

Proof

[Ex.]

This topology is known as the Zariski topology of V.
For each of these open sets $\mathcal{U} \subseteq V$ we have a commutative ring

$$\mathcal{O}(\mathcal{U}) := \left\{ \frac{f}{g} \in \overline{K}(V) \;\middle|\; \mathcal{U} \subseteq D_g \right\}.$$

In order to unravel the nature of the assignation

$$\mathcal{U} \longmapsto \mathcal{O}(\mathcal{U})$$

we shall discuss the basics of category theory, as follows.