Lecture 2 Some key structures

A category $\mathcal{C}$ consists of a class of objects $\mathrm{ob}(\mathcal{C})$, $\forall A, B \in \mathrm{ob}(\mathcal{C})$ a set $\mathrm{Mor}(A,B)$ whose elements are denoted $A \xrightarrow{f} B$, and $\forall A, B, C \in \mathrm{ob}(\mathcal{C})$

a map

$$\mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) \longrightarrow \mathrm{Mor}(A,C)$$

$$(A \xrightarrow{f} B,\ B \xrightarrow{g} C) \longmapsto A \xrightarrow{g \circ f} C$$

such that

(i) $\forall A \in ob(\mathcal{C})\ \exists\ e_A \in Mor(A, A)$ s.t.

$$\forall f \in Mor(A, B): \quad f \circ e_A = f \quad \text{and} \quad e_B \circ f = f.$$

(ii) $\forall f \in Mor(A, B),\ g \in Mor(B, C),\ h \in Mor(C, D)$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

<u>Remark</u> It is easy to see that $\forall A \in ob(\mathcal{C})$, if e_A and e_A' are as in (i), then $e_A = e_A'$.

We usually denote this element as 1_A.

__Example__ The class of all groups are the objects of the category Grp whose morphisms are the group homomorphisms.

__Example__ Given a topological space X, the open sets U of X are the objects of a category $\mathcal{C}_X$ whose morphisms are the inclusion maps.

Given categories $\mathcal{C}$ & $\mathcal{D}$, a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is defined by

$$\begin{aligned} ob(\mathcal{C}) &\xrightarrow{F} ob(\mathcal{D}) \\ A &\longmapsto F(A) \end{aligned}$$

and by

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(A, B) &\xrightarrow{F} \mathrm{Mor}_{\mathcal{D}}(F(A), F(B)), \\ \left(A \xrightarrow{f} B\right) &\longmapsto \left(F(A) \xrightarrow{F(f)} F(B)\right) \end{aligned}$$

$\forall\, A, B \in ob(\mathcal{C})$, s.t. $\forall\, f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(B, C)$:

$$\left(\begin{tikzcd} C & \\ & B \arrow[ul, "g"'] \\ A \arrow[uu, "g \circ f"] \arrow[ur, "f"'] & \end{tikzcd}\right) \longmapsto \left(F(g \circ f) = F(g) \circ F(f) \quad \begin{tikzcd} F(C) & \\ & F(B) \arrow[ul, "F(g)"'] \\ F(A) \arrow[uu] \arrow[ur, "F(f)"'] & \end{tikzcd}\right)$$

<u>Example</u> Let $n \in \{1, 2, 3, \dots\}$. We have a functor

$$\mathrm{CRng} \xrightarrow{GL_n} \mathrm{Grp}$$

$$\left(\begin{array}{c} B \\ f \uparrow \\ A \end{array} \right) \longmapsto \left(\begin{array}{cc} GL_n(B) & \begin{pmatrix} f(a_{11}) & \dots & f(a_{1n}) \\ \vdots & & \\ f(a_{n1}) & \dots & f(a_{nn}) \end{pmatrix} \\ GL_n(f) \uparrow & \rotatebox{90}{$\mapsto$} \\ GL_n(A) & \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{n1} & & a_{nn} \end{pmatrix} \end{array} \right)$$

Example We have another functor

$$\mathrm{CRng} \xrightarrow{\times} \mathrm{Grp}$$

$$\left(\begin{array}{c} B \\ f \uparrow \\ A \end{array} \right) \longmapsto \left(\begin{array}{c} B^\times \\ f^\times \uparrow \\ A^\times \end{array} \right)$$

Ex. Let $\det(M)$ denote the determinant of $M \in M_n(A)$, where $M_n(A)$ denotes the ring of n by n matrices with entries in a ring A. Then

$$
\begin{array}{ccc}
\mathbf{CRng} & & \mathbf{Grp} \\[1em]
\begin{tikzcd}
B \\
A \arrow[u, "f"]
\end{tikzcd}
& \Longrightarrow &
\begin{tikzcd}
GL_n(B) \arrow[r, "\det"] & B^\times \\
GL_n(A) \arrow[u, "GL_n(f)"] \arrow[r, "\det"] \arrow[ur, phantom, "\circlearrowright"] & A^\times \arrow[u, "f^\times"']
\end{tikzcd}
\end{array}
$$

Def'n Given functors $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$, a natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\tau"'] \end{tikzcd}$$

is a map

$$ob(\mathcal{C}) \xrightarrow{\tau} Mor_{\mathcal{D}}(F(A), G(A))$$

$$A \longmapsto \tau_A$$

s.t.

$$\mathcal{C} \qquad\qquad\qquad\qquad \mathcal{D}$$

$$\begin{tikzcd} B \\ A \arrow[u, "f"] \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} F(B) \arrow[r, "\tau_B"] & G(B) \\ F(A) \arrow[u, "F(f)"] \arrow[r, "\tau_A"] \arrow[ur, phantom, "\circlearrowright"] & G(A) \arrow[u, "G(f)"'] \end{tikzcd}$$

Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine varieties, and suppose that the latter is the locus of zeros of given polynomials $g_1, \dots, g_k \in \bar{K}[X_1, \dots, X_m]$. A morphism $V \xrightarrow{\psi} W$ is $\psi = (\psi_1, \dots, \psi_m) \in \underbrace{\bar{K}[V] \times \cdots \times \bar{K}[V]}_{m\text{-times}}$ s.t.

$$\begin{cases} g_1(\psi_1, \dots, \psi_m) = 0, \\ \quad \vdots \\ g_k(\psi_1, \dots, \psi_m) = 0. \end{cases}$$

The ring $\overline{K}[V]$ is known as the coordinate ring of V, as n-tuples of its elements define the morphisms in this category $\mathcal{V}_{\overline{K},\mathrm{aff}}$ of affine algebraic varieties over $\overline{K}$.

<u>Example</u> Let $V = \mathbb{A}^2$ and $W \subseteq \mathbb{A}^3$ be the locus of zeros of $g(X_1, X_2, X_3) = X_1^2 + X_2^2 - X_3^2$. Then

$$\psi = (\psi_1, \psi_2, \psi_3) = (2Y_1Y_2, Y_1^2 - Y_2^2, Y_1^2 + Y_2^2)$$

defines a morphism

$$\psi : V \longrightarrow W$$

as obviously $\psi_1^2 + \psi_2^2 - \psi_3^2 = 0$, regarded an identity in the ring $\bar{K}[V] = \bar{K}[Y_1, Y_2]$.

Each morphism $V \xrightarrow{\psi} W$ induces a ring homomorphism

$$\bar{K}[W] \xrightarrow{\psi^*} \bar{K}[V]$$

$$f \longmapsto f \circ \psi$$

It turns out that $\psi \longmapsto \psi^*$ defines a functor from the category $\mathcal{V}_{\bar{K}, \mathrm{aff}}$ of affine varieties over $\bar{K}$ to $\mathrm{CRng}^{\mathrm{op}}$. (Further details soon.)

Remark The above construction is formally similar with the dual space construction of basic linear algebra. Indeed, given a field F and an F-linear map $V \xrightarrow{L} W$, the dual L^* of L is the F-linear map

$$\begin{aligned} W^* &\xrightarrow{L^*} V^* \\ \ell &\longmapsto \ell \circ L \end{aligned},$$

where $V^* := L_F(V, F)$ with its natural F-vector space structure.

<u>Defn</u> Let X be a topological space. Let $\mathcal{E}_X$ be the category we attached to X. A presheaf on X of commutative rings is a functor

$$\mathcal{E}_X \xrightarrow{\mathcal{F}} \mathrm{CRng}^{op},$$

$$\begin{pmatrix} U \\ | \\ V \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{F}(U) & & f \\ \downarrow & \mathcal{F}_{U,V} & \rotatebox{90}{$\mapsto$}\!\!\!\downarrow \\ \mathcal{F}(V) & & f|V \end{pmatrix}$$

<u>Def'n</u> We say that a presheaf $\mathcal{F}$ on a topological space X is a sheaf if for each open set $\mathcal{U}$ of X and each open covering $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of $\mathcal{U}$:

(i) $\forall\, f, g \in \mathcal{F}(\mathcal{U})$: (locality axiom)

$$\left(\forall \alpha \in I : f|\mathcal{U}_\alpha = g|\mathcal{U}_\alpha \right) \Rightarrow f = g .$$

(ii) $\forall\, \{ f_\alpha \in \mathcal{F}(\mathcal{U}_\alpha) \}_{\alpha \in I}$: (gluing axiom)

$$\left(\forall \alpha, \beta \in I : f_\alpha | \mathcal{U}_\alpha \cap \mathcal{U}_\beta = f_\beta | \mathcal{U}_\alpha \cap \mathcal{U}_\beta \right) \Rightarrow \exists f \in \mathcal{F}(\mathcal{U}) : \forall \alpha \in I :$$

$$f|\mathcal{U}_\alpha = f_\alpha .$$

__Defn__ Given sheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X a morphism $\mathcal{F} \xrightarrow{\mu} \mathcal{G}$ is a natural transformation

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, bend left, "\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathrm{CRng}^{op} \end{tikzcd}$$

__Remark__ Sheaf theory was discovered by Jean Leray while he was prisioner of war in WW II in Edelbach.