Lecture 3 Geometry of projective space

The projective n- space $\mathbb{P}^n$ is the quotient set

$$\mathbb{P}^n := \mathbb{K}^{n+1} \setminus \{0\} \,/\!\sim$$

where

$$(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$$

$$\Updownarrow$$

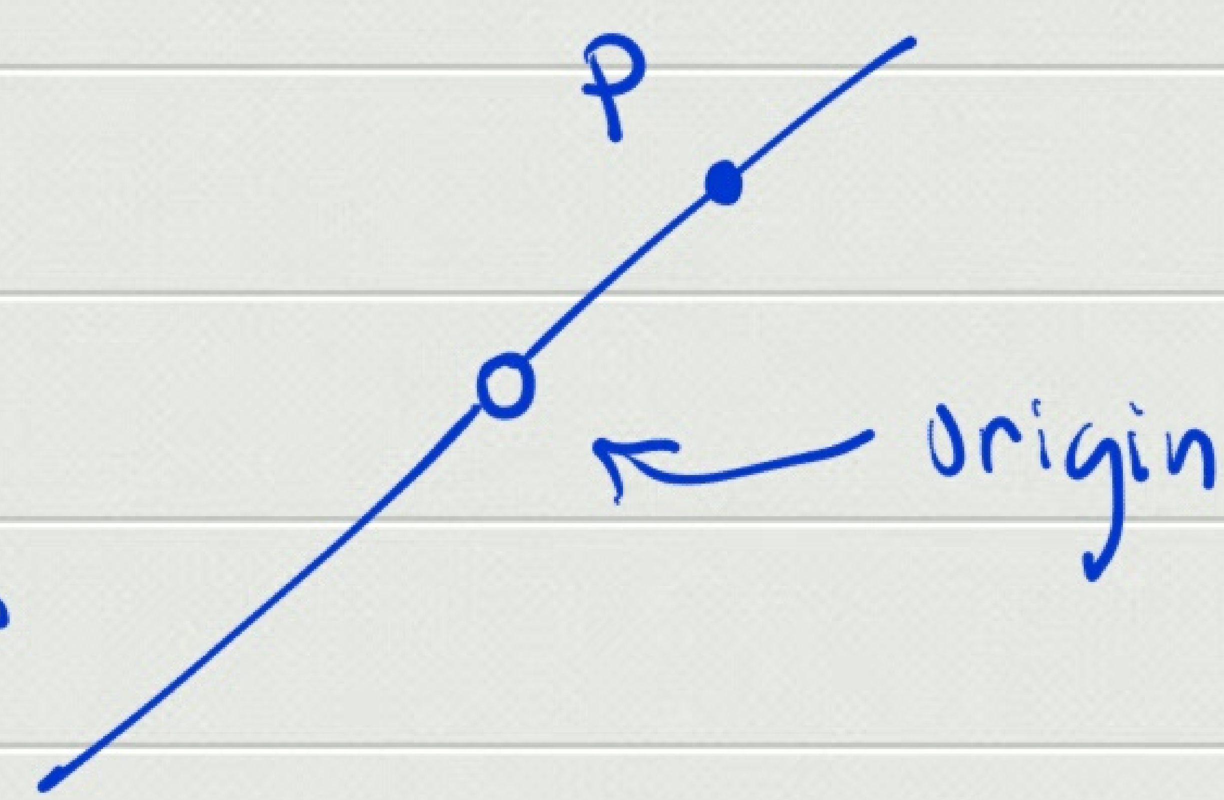
$$\exists\, \lambda \in \mathbb{K}^\times \text{ s.t. } (x_0, x_1, \dots, x_n) = \lambda\, (y_0, y_1, \dots, y_n)$$

The relation $\sim$ is indeed an equivalence relation (the verification is left to the reader), so $\mathbb{P}^n$ is well-defined. We denote

$$(x_0 : x_1 : \dots : x_n) \in \mathbb{P}^n$$

the equivalence class of $P = (x_0, x_1, \dots, x_n) \in \overline{K}^n \setminus \{0\}$

$(x_0 : x_1 : \dots : x_n) \longrightarrow$



Note that $\mathbb{P}^n$ is thus a set of punctured lines.

Our first aim is to parametrise some subsets of these sets of punctured lines $\mathbb{P}^n$. The first of these parametrisations is

$$\mathbb{A}^n \xrightarrow{\nu_1} \mathbb{P}^n$$

$$(x_1, x_2, \dots, x_n) \longmapsto (1 : x_1 : x_2 : \dots : x_n)$$

which parametrises all these punctured lines, except for those of the form $(0 : x_1 : \dots : x_n)$, where $(x_1 : x_2 : \dots : x_n) \in \mathbb{P}^{n-1}$.

Similarily, also

$$\mathbb{A}^n \xrightarrow{\nu_2} \mathbb{P}^n$$

$$(x_1, \ldots, x_n) \longmapsto (x_1 : 1 : x_2 : \ldots : x_n)$$

is injective and the complement of its image is the set

$$\{(x_1 : 0 : x_2 : \ldots : x_n) \in \mathbb{P}^n \mid (x_1 : x_2 : \ldots : x_n) \in \mathbb{P}^{n-1}\}.$$

This way we may construct injective maps

$$\mathbb{A}^n \xrightarrow{\nu_k} \mathbb{P}^n \qquad (k = 1, \ldots, n+1)$$

s.t.

$$\nu_1(\mathbb{A}^n) \cup \ldots \nu_{n+1}(\mathbb{A}^n) = \mathbb{P}^n.$$

Defn The Zariski topology of $\mathbb{P}^n$ is the one whose closed sets are of the form

$$\{ (x_0 : x_1 : \dots x_n) \in \mathbb{P}^n \text{ s.t. } \\ g_1(x_0, x_1, \dots, x_n) = 0 \\ \vdots \\ g_m(x_0, x_1, \dots, x_n) = 0 \},$$

where $g_1, \dots, g_m \in \bar{K}[X_0, \dots X_n]$ are homogeneous.

We may see that $\{\mathcal{U}_1, \dots, \mathcal{U}_{n+1}\}$, where

$$\mathcal{U}_k := \nu_k(\mathbb{A}^n) \qquad (k = 1, \dots, n+1)$$

is a Zariski open cover of $\mathbb{P}^n$ and, moreover,

$$\mathcal{U}_k \cong \mathbb{A}^n \qquad (k = 1, \dots, n+1).$$

So $\mathbb{P}^n$ is locally homeomorphic to $\mathbb{A}^n$.