Lecture 4 Examples

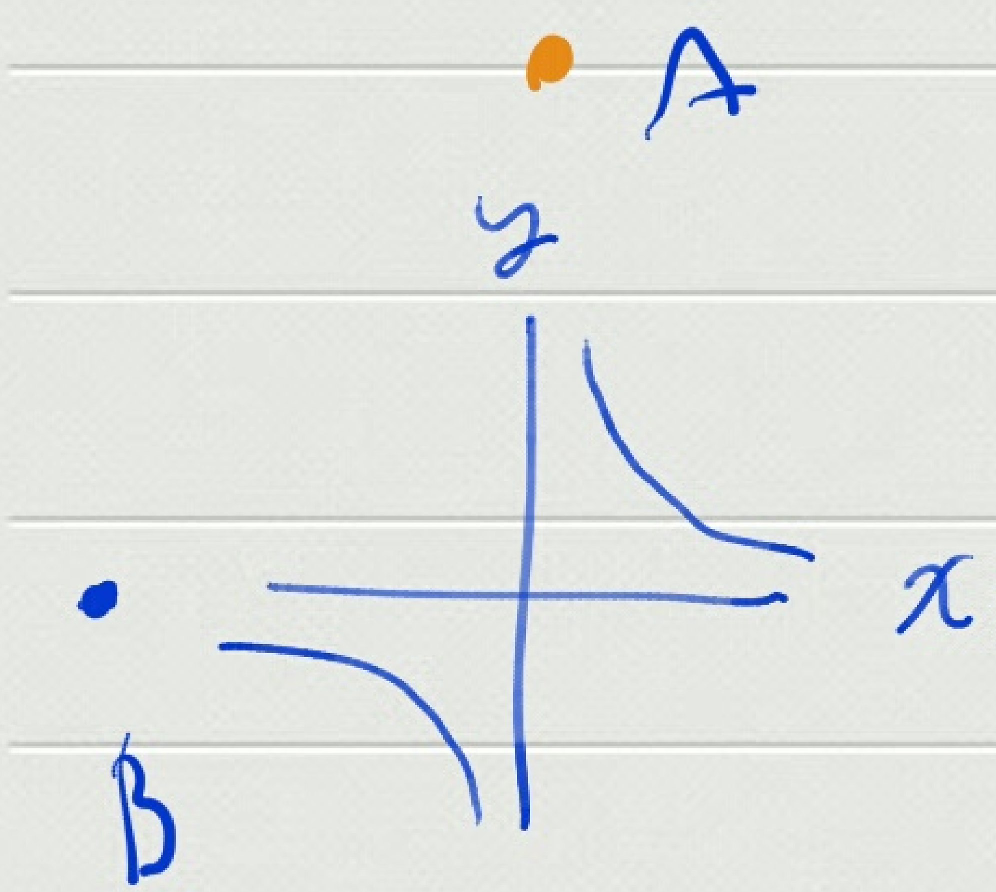
$$F^{hom} = XY - Z^2$$

$F = XY - 1$ $\qquad\qquad$ $V^{proj}_{F^{hom}}$ $\qquad\qquad$ $G = X - Z^2$

$$V_{F} \subseteq \mathbb{A}^2 \xrightarrow{\ \nu_3\ } \mathbb{P}^2 \xleftarrow{\ \nu_2\ } \mathbb{A}^2$$

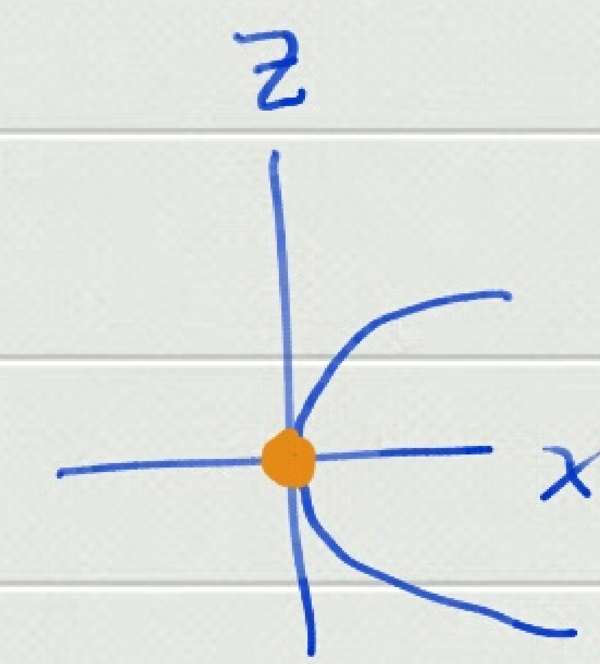
(with $V^{proj}_{F^{hom}} \subseteq \mathbb{P}^2$)

$$(x, y) \mapsto (x : y : 1) \qquad\qquad (\alpha : 1 : \gamma) \longleftarrow\!\shortmid (\alpha, \gamma)$$



$$D_y \longrightarrow \nu_3(\mathbb{A}^2) \cap \nu_2(\mathbb{A}^2) \longrightarrow \mathbb{A}^2$$

$$(\alpha : \beta : \gamma) \longmapsto \left(\frac{\alpha}{\beta}, \frac{\gamma}{\beta}\right)$$



$$(x, y) \longmapsto \left(\frac{x}{y}, \frac{1}{y}\right)$$

$$\varphi$$

Now consider the curve given by the Weierstraß eq'n

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

where $a_1, a_2, a_3, a_4, a_6 \in k$. Consider the change of coordinates

$$\begin{cases} x = \frac{z}{w} \\ y = -\frac{1}{w} \end{cases}$$

Then

$$\frac{1}{w^2} - a_1 \frac{z}{w^2} - a_3 \frac{1}{w} = \frac{z^3}{w^3} + a_2 \frac{z^2}{w^2} + a_4 \frac{z}{w} + a_6$$

$$w - a_1 zw - a_3 w^2 = z^3 + a_2 z^2 w + a_4 z w^2 + a_6 w^3$$

$$w = \underbrace{z^3 + a_1 zw + a_2 z^2 w + a_3 w^2 + a_4 z w^2 + a_6 w^3}_{\overset{\shortparallel}{f(z,w)}}$$

In terms of the new variables, the cubic (★) is thus

$$w = f(z, w).$$

*

Now consider the sequence $f_1, f_2, f_3, \ldots \in R[z, w]$

$$f_1 := f(z, w),$$

$$f_2 := f_1(z, f(z, w)),$$

$$f_3 := f_2(z, f(z, w)),$$

$$\vdots$$

where $R = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$.

Th'm We have

$$w(z) := \lim_{n\to\infty} f_n(z,0) \in R[[z]].$$

Moreover, $\big(z, w(z)\big) \in \mathbb{A}^2(S)$, where $S = R[[z]]$, is the only S-rational point of the cubic $(*)$. Actually,

$$w(z) = z^3\big(1 + A_1 z + A_2 z^2 + \cdots\big),$$

where $A_i \in \mathbb{Z}[a_1, \ldots, a_6]$, homogeneous of weight i, where $wt(a_i) := i$.

Explicitly, we have

$$w(z) = z^3 \Big[1 + a_1 z + (a_1^2 + a_2) z^3 + (a_1^3 + 2a_1 a_2 + a_3) z^3 + (a_1^4 + 3a_1^2 a_2 + 3a_1 a_3 + a_4) z^4 + \dots \Big]$$

Via SageMath:

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P.<a1, a2, a3, a4, a6>
E = EllipticCurve(list(P.gens))
E.formal_group().w(10)
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We express this in terms of the original variables and get

$$x(z) = \frac{z}{w(z)} = \frac{1}{z^2} - \frac{a_1}{z} - a_2 - a_3 z - (a_4 + a_1 a_3) z^2 + \dots$$

$$y(z) = -\frac{1}{w(z)} = -\frac{1}{z^3} + \frac{a_1}{z^2} \dots$$

$$\omega(z) = \frac{dx}{y} = \left(1 + a_1 z + \dots\right) dz$$

If $\chi(k) = 0$, then we may use some classical complex-analytic methods, as we shall now sketch. Let $\omega_1, \omega_2 \in \mathbb{C}^\times$ s.t. $\tau =: \omega_1/\omega_2 \in \mathcal{H}$ and define $\forall\, z \in \mathbb{C} - \Lambda$,

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

where $\Lambda := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$.

Thm If we write $x = \wp_\Lambda(z)$ and $y = \wp_\Lambda'(z)$, then

$$y^2 = 4x^3 - 60\, G_4\, x - 140\, G_6 ,$$

where

$$G_n = \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^n} ,$$

for each $n \in \{4, 6, 8, 10, \dots\}$.

In particular, we have a map

$$\mathbb{C}/\Lambda \xrightarrow{\sim} V_F^{proj} \subseteq \mathbb{P}^2(\mathbb{C}),$$

$$z \longmapsto \begin{cases} (\wp_\Lambda(z) : \wp'_\Lambda(z) : 1), & \text{if } z \notin \Lambda \\ O_E, & \text{if } z \in \Lambda \end{cases}$$

where

$$F = y^2 - x^3 + 60 G_4 x + 140 G_6 .$$

We have the Laurent expn

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)\, G_{2n+2}\, z^{2n}$$

around $z = 0$.