Lecture 5 Localisation

<u>Defn</u> Given a commutative ring R, we say that $S \subseteq R$ is a *multiplicative subset* of R if it is a multiplicative submonoid of R.

<u>Example</u> If $p \subseteq R$ is a prime ideal of R, then $S_p := R \setminus p$ is a multiplicative subset of R.

<u>Example</u> Let $f \in R$. Then $S_f := \{1, f, f^2, \dots\}$ is a multiplicative subset of R.

Given a commutative ring R and a multiplicative subset $S \subseteq R$, the localisation $R[S^{-1}]$ is the quotient set

$$R[S^{-1}] := (R \times S) / \sim ,$$

when $(r, s) \sim (r', s')$ iff $\exists\, s_1 \in S$ s.t.

$$s_1 (r s' - s' r) = 0.$$

It is easy to see that $\sim$ is indeed an eq'c relation.

We have the natural map

$$R \times S \xrightarrow{\pi} (R \times S)/\sim$$

$$(r,s) \longmapsto \pi(r,s) =: \frac{r}{s}$$

Moreover, $R[S^{-1}]$ has a natural ring structure:

$$\frac{r}{s} + \frac{r'}{s'} := \frac{rs' + r's}{ss'}, \qquad \frac{r}{s} \cdot \frac{r'}{s'} := \frac{rr'}{ss'}.$$

It is easy to see that we have a ring homomorphism

$$\begin{array}{ccccc} \mathfrak{p} \subseteq & R & \xrightarrow{\varphi} & R[S^{-1}] & \\ & r & \longmapsto & \frac{r}{1} & \rotatebox{90}{$\subseteq$}\ \varphi(\mathfrak{p}) \end{array}$$

with kernel

$$\ker(\varphi) = \{ r \in R \mid \exists\, s \in S \text{ s.t. } sr = 0 \}.$$

Note that if R is an integral domain, then the ring homomorphism φ is an embedding, provided $0 \notin S$.

<u>Def'n</u> If $\mathfrak{p}$ is a prime ideal of R, then we define the localization of R at $\mathfrak{p}$ as

$$R_{\mathfrak{p}} := R[S_{\mathfrak{p}}^{-1}],$$

where $S_{\mathfrak{p}} = R \setminus \mathfrak{p}$, as before.

<u>Def'n</u> We say that a commutative ring R is a local ring if it has exactly one maximal ideal $\mathfrak{m}$.

<u>Example</u> Given any R and $\mathfrak{p}$ as above, the localization $R_{\mathfrak{p}}$ is a local ring. Its maximal ideal $\mathfrak{m}_{\mathfrak{p}}$ is given by

$$\mathfrak{m}_{\mathfrak{p}} := \mathfrak{p} R_{\mathfrak{p}}$$

Then

$$\kappa(\mathfrak{p}) := R_{\mathfrak{p}} / \mathfrak{m}_{\mathfrak{p}}$$

is a field which is known as the residue field at $\mathfrak{p}$.

Let

$$\mathrm{Spec}(R) = \{ \mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal} \}.$$

Given a field K, we say that $\mathfrak{p} \in \mathrm{Spec}(R)$ is K-rational if $\kappa(\mathfrak{p}) \cong K$.

Another example of localization is obtained by picking $f \in R$ and defining

$$R_f := R[S_f^{-1}],$$

where $S_f = \{1, f, f^2, \dots\}$, as before.