Lecture 7 Differential forms

Define the cotangent space of $\mathcal{U}$ at p as the $\overline{K}$ - vector space

$$T_p^*(\mathcal{U}) = \mathfrak{m}_p / \mathfrak{m}_p^2 ,$$

where

$$\mathfrak{m}_p := \{ f \in \mathcal{O}_V(\mathcal{U}) \mid f(p) = 0 \} .$$

Remark Later we shall prove that $\mathfrak{m}_p$ is a maximal ideal of $\mathcal{O}_V(\mathcal{U})$.

We have a natural map

$$\mathcal{O}(\mathcal{U}) \longrightarrow T_p^*(\mathcal{U})$$

$$f \longmapsto \underbrace{f - f(p)}_{\in\, \mathfrak{m}_p} + \mathfrak{m}_p^2 =: (df)_p$$

<u>Defn</u> We call $(df)_p$ the differential of f at p.

<u>Defn</u> We say that $\mathcal{U}$ is non-singular at $p \in \mathcal{U}$ if

$$\dim_{\bar{K}} T_p^*(\mathcal{U}) = \dim(\mathcal{U}).$$

If $\mathcal{U}$ is non-singular at every $p \in \mathcal{U}$ then we say that $\mathcal{U}$ is non-singular.

Suppose $\mathcal{U}$ is non singular. For $p \in \mathcal{U}$ $\exists$ $t_1, \dots, t_n \in \mathfrak{m}_p$ s.t. $(dt_1)_p, \dots, (dt_n)_p$ is a basis for $T_p^*(\mathcal{U})$.

Define

$$T^*(\mathcal{U}) := \dot{\bigcup_{p \in \mathcal{U}}} T_p^*(\mathcal{U})$$

and call it the cotangent bundle / $\mathcal{U}$. It comes equipped with a canonical projection map

$\pi \circ s = 1_{\mathcal{U}}$

$$\begin{tikzcd} T^*(\mathcal{U}) \arrow[d, "\pi"'] \\ \mathcal{U} \arrow[u, bend left=60, "s"] \end{tikzcd} \qquad \begin{tikzcd} (p, (df)_p) \arrow[d, mapsto] \\ p \end{tikzcd}$$

Def'n A differential 1-form ω on $\mathcal{U}$ is a section of the above canonical projection map.

Example We have the differential form ω defined by
$\omega(p) := (df)_p$, $f \in \mathcal{O}(\mathcal{U})$. We write $\omega =: df$.

<u>Remark</u> Not all differential 1-forms ω on $\mathcal{U}$ are necessarily of the form $\omega = df$, for some $f \in \mathcal{O}(\mathcal{U})$. These are known as exact differential forms.

If $\omega = df$, as above and $f \in \mathcal{O}(\mathcal{U})$ then

$$p \mapsto (g\,df)(p) := g(p)\,(df)_p$$

is a differential 1-form. We'll denote it $g\,df$.

Example Consider the curve $V \subseteq \mathbb{A}^2$ attached to

$$F = X^3 + X - Y^2 \qquad (\star)$$

Note that $P = (0,0) \in V$ and

$$\mathfrak{m}_P = \langle x, y \rangle \text{ and } \mathfrak{m}_P^2 = \langle x^2, xy, y^2 \rangle,$$

where $x := \overline{X}$ and $y := \overline{Y}$ in $\overline{K}[V]$. But $(\star)$ implies that

$$x = y^2 - x^3 \equiv 0 \pmod{\mathfrak{m}_P^2},$$

so $(dx)_P = 0$ and thus $T_P^* = \overline{K} \cdot (dy)_P$.

We call $\{t_1, \ldots, t_n\}$ a set of *local parameters* at P.

<u>Thm</u> The maximal ideal $\mathfrak{m}_P$ a nonsingular point P is generated by the set of local parameters, that is

$$\mathfrak{m}_P = \langle t_1, \ldots, t_n \rangle$$

Proof

[Later — via Nakayama's lemma.]

<u>Th'm</u> Let $P \in \mathcal{U}$, where $\mathcal{U} \subseteq V$ is open. Suppose P is non singular and let $t_1, \ldots, t_n$ be a system of local parameters at P. If ω is a differential 1-form on $\mathcal{U}$, then $\exists\ g_1, \ldots, g_n \in \mathcal{O}_V(\mathcal{U})$ s.t. $\forall Q \in \mathcal{U}$

$$\omega(Q) = g_1(Q)(dt_1)_Q + \cdots + g_n(Q)(dt_n)_Q.$$

Here n denotes the dimension of V.