Lecture 8 The divisor of a differential form on a curve

Let C be a nonsingular projective curve and let $P \in C$.
So we may pick a local parameter t at P, i.e.
$t \in \mathfrak{m}_P \subseteq \mathcal{O}_{V,P}(\mathcal{U})$, for some Zariski open neighbourhood $\mathcal{U}$ of P and dt generates the 1-dimensional $\overline{K}$-vector space

$$T_P^*(C) = \mathfrak{m}_P / \mathfrak{m}_P^2$$

Prop'n Let C, P, and t be as above. Then for each differential 1-form ω there exists a unique $f \in \overline{K}(C)$ s.t.

$$\omega = f\, dt.$$

Remark The above prop'n says that the $\overline{K}$-vector space of Kähler differentials Ω^1_C of C is in fact a 1-dimensional $\overline{K}(C)$-vector space.

Defn With the above assumptions, the order of $\omega \neq 0$ at P is

$$\operatorname{ord}_P(\omega) := \operatorname{ord}_P(f) \in \mathbb{Z},$$

and the divisor of a non zero differential 1-form is

$$(\omega) := \sum_{P \in C} \operatorname{ord}_P(\omega)\, e_P .$$

We say that such ω is *holomorphic* if either $\omega = 0$ or

$$(\omega) \geqslant 0.$$

Example Suppose $C = \mathbb{P}^1$. Consider the element

$$\omega = dX \in \Omega^1_C ,$$

where X denotes the "indeterminate" in the polynomial ring $\bar{K}[X]$. Then $(\omega) = 2(e_0 - e_\infty)$ and the only holomorphic differential 1-form is $\omega = 0$.

Ex. Prove these two assertions.

Example Suppose that C is the projective curve given by (the homogeneous polynomial attached to)

$$F = Y^2 - (X-\alpha_1)(X-\alpha_2)(X-\alpha_3),$$

where $\alpha_1, \alpha_2, \alpha_3 \in \overline{K}$ with $\chi(K) \neq 2, 3$.

Then

$$\omega := \frac{dX}{Y}$$

is holomorphic.