Lecture 9 The Noetherian chain condition and primary decomposition

Let R be a commutative ring. We say that R is *Noetherian* if every strictly ascending chain of ideals of R

$$I_1 \subsetneq I_2 \subsetneq \cdots$$

terminates. The dual notion is that of an *Artinian* ring.

Remark We'll show a little later that $R = \mathbb{Z}$ is Noetherian. But it is clearly not Artinian.

<u>Prop'n</u> Let R be any commutative ring. The following are equivalent.

(a) $\forall$ ideal $I \subseteq R$ is s.t. $I = \langle r_1, \dots, r_m \rangle$

(b) R is Noetherian

(c) Every non empty set of ideals has a maximal element.

Proof

(a) $\Rightarrow$ (b): Let $I_1 \subseteq I_2 \subseteq \cdots$ be any chain of ideals of R.

Then

$$I := \bigcup_{n=1}^{\infty} I_n \qquad \star$$

is an ideal of R, so $I = \langle r_1, \dots, r_m \rangle$, for some $r_1, \dots, r_m \in I$.

So $\exists\ n_0 \in \mathbb{Z}_{>0}$ s.t. $\{r_1, \ldots, r_m\} \subseteq I_{n_0}$.

Thus $I_{n_0} = I_{n_0+1} = \cdots$ and (b) follows.

(b) $\Rightarrow$ (c): Suppose that $\mathcal{S}$ is a nonempty set of ideals that has no maximal element. So $\exists\ I_1 \in \mathcal{S}$.

<u>Claim</u> There is $I_2 \in \mathcal{S}$ s.t. $I_1 \subsetneq I_2$.

Proof of claim

[Ex.] $\square$

From the claim we obtain an infinite chain of ideals of R,

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$$

Thus (c) follows.

(c) $\Rightarrow$ (b): The maximal chain condition applied to a set of ideals $\mathcal{S} = \{ I_1, I_2, \dots \}$ s.t. $I_1 \subseteq I_2 \subseteq \dots$ yields (b).

(b) $\Rightarrow$ (a): Assume $\exists$ a non finitely generated ideal $I \subseteq R$.

So $\forall \{ r_1, \dots, r_m \} \subseteq I$ we have $\langle r_1, \dots, r_m \rangle \subsetneq I$.

Thus $\exists\, r_{m+1} \in I$ s.t. $r_{m+1} \notin \langle r_1, \dots, r_m \rangle$. It follows that we may construct inductively a chain of ideals of R,

$$\langle r_1 \rangle \subsetneq \langle r_1, r_2 \rangle \subsetneq \langle r_1, r_2, r_3, \dots \rangle \subsetneq \dots$$

Hence (a) $\square$

Given any commutative ring R, its *Krull dimension* is the supremum of the lengths n of all prime ideal chains

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

in $\mathrm{Spec}(R)$. This is the key invariant attached to certain Noetherian rings R such as $R = K[V]$, where V is an algebraic set.

Another key object attached to a Noetherian ring R is the *primary decomposition*

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_g \qquad \star$$

of any given ideal $\mathfrak{a} \subseteq R$, where $\mathfrak{q}_1, \ldots, \mathfrak{q}_g \subseteq R$ are ideals s.t.

$$\mathrm{rad}(\mathfrak{q}_i) =: \mathfrak{p}_i \in \mathrm{Spec}(R);$$

we say that $\mathfrak{q}_i$ is $\mathfrak{p}_i$*-primary*, $\forall\, i \in \{1, \ldots, g\}$. We say that $(\star)$ is *minimal* if g is minimal and $\mathfrak{p}_i \neq \mathfrak{p}_j$. If so, the subset

$$\mathrm{Ass}(\mathfrak{a}) := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_g\} \subseteq \mathrm{Spec}(R)$$

is uniquely determined. We also define

$$\mathrm{Ass}'(\mathfrak{a}) := \{\, \mathfrak{p} \in \mathrm{Ass}(\mathfrak{a}) \mid \mathfrak{p} \text{ is minimal} \,\}$$

and its members are known as the *isolated* primes of the set of associated primes of the ideal $\mathfrak{a} \subseteq R$.