Lecture 10 The Hilbert basis theorem and Krull's Hauptidealsatz

Th'm (Hilbert basis) If R is Noetherian, then so is $R[X]$.

Corollary If K is a field then the polynomial ring in n variables $R = K[X_1, X_2, \ldots, X_n]$ is Noetherian, so

$$P = (x_1, \ldots x_n) \in V(I) \Longleftrightarrow \begin{cases} r_1(x_1, \ldots, x_n) = 0 \\ \vdots \qquad\qquad\qquad \vdots \\ r_m(x_1, \ldots, x_n) = 0 \end{cases}$$

where $r_1, \ldots, r_m \in I$ are s.t. $I = \langle r_1, \ldots, r_m \rangle$.

Proof of thm — as simplified by Heidrun Sarges *

Suppose that there is a non finitely generated ideal $J \subseteq R[X]$. Then

$$
\begin{array}{lll}
\vdots & \vdots & \vdots \\
J_2 = \langle f_0, f_1, f_2 \rangle, & f_2 = \overset{0}{\overset{\#}{a_2}} X^{n_2} + \ldots \in J - J_1, & \min n_2 \\
\cup\!\!\!\!\!\!\neq & & \\
J_1 = \langle f_0, f_1 \rangle, & f_1 = \overset{0}{\overset{\#}{a_1}} X^{n_1} + \ldots \in J - J_0, & \min n_1 \\
\cup\!\!\!\!\!\!\neq & & \\
J_0 = \langle f_0 \rangle, & f_0 = 0 &
\end{array}
$$

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* Sarges, H., *Ein Beweis des Hilbertschen Basissatz*, Journal für die reine und angewandte Mathematik, 283/284, (1976), pp 436-437.

But R is Noetherian, so $\exists\ k \in \{1, 2, 3, \dots\}$ s.t.

$$
\begin{array}{l}
I_{k+1} = \langle a_1, \dots, a_k, a_{k+1} \rangle \\
\parallel \\
I_k = \langle a_1, \dots, a_k \rangle \\
\vdots \\
\rotatebox{90}{$\subsetneq$} \\
I_2 = \langle a_1, a_2 \rangle \\
\rotatebox{90}{$\subsetneq$} \\
I_1 = \langle a_1 \rangle
\end{array}
$$

In particular, $\exists\ b_1, \dots, b_k \in R$ s.t. $b_1 a_1 + \dots b_k a_k = a_{k+1}$.

Therefore

$$g := b_1 X^{n_{k+1} - n_1} \overbrace{\left(a_1 X^{n_1} + \dots \right)}^{f_1} +$$

$$b_2 X^{n_{k+1} - n_2} \overbrace{\left(a_2 X^{n_2} + \dots \right)}^{f_2} +$$

$$\vdots$$

$$b_k X^{n_{k+1} - n_k} \overbrace{\left(a_k X^{n_k} + \dots \right)}^{f_k}$$

$$= \left(b_1 a_1 + \dots + b_k a_k \right) X^{n_{k+1}} + \dots = a_{k+1} X^{n_{k+1}} + \dots$$

$\in \langle f_1, \dots, f_k \rangle$ and $h := f_{k+1} - g$ is s.t.

$$\partial h < \partial f_{k+1}$$

but $h \notin \langle f_1, \ldots, f_k \rangle$ as $f_{k+1} \notin \langle f_1, \ldots, f_k \rangle$.

This contradicts the minimality of the degree of f_{k+1}.

Hence $R[X]$ is Noetherian $\square$

The *height* $ht(\mathfrak{p})$ of $\mathfrak{p} \in Spec(R)$ is the supremum of the lenths of ℓ of chains

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_\ell \in Spec(R)$$

s.t. $\mathfrak{p}_\ell = \mathfrak{p}$. In other words, $ht(\mathfrak{p}) = \dim(R_\mathfrak{p})$.

Theorem (Krull) If R is Noetherian and $\mathfrak{a} \subseteq R$ is an ideal generated by $a_1, \ldots, a_r \in \mathfrak{a}$, then for each minimal prime divisor $\mathfrak{p}$ of $\mathfrak{a}$:

$$ht(\mathfrak{p}) \leq r$$

Proof

[It shall be added here in due course.]

The Jacobson radical $j(R)$ of a ring R is the intersection

$$j(R) = \bigcap_{\text{maximal } \mathfrak{m} \in \operatorname{Spec}(R)} \mathfrak{m}$$

<u>Lemma</u> For each $a \in R$ the following are equivalent

(i) $a \in j(R)$

(ii) $\forall\, b \in R : 1 - ab \in R^\times$

Proof

(i) $\Rightarrow$ (ii): Note that $\overline{1-ab} = \overline{1}$ in $R/\mathfrak{m}$, so $1 - ab \notin \mathfrak{m}$, for each maximal ideal $\mathfrak{m} \subseteq R$. But every non-unit is contained in a maximal ideal. Thus (ii).

$\neg(i) \Rightarrow \neg(ii)$: So $a \notin R \setminus j(R)$, i.e. $\exists$ maximal $\mathfrak{m} \in \operatorname{Spec}(R)$ s.t. $a \notin \mathfrak{m}$.

Therefore $\langle \mathfrak{m}, a \rangle_R = R$, so $1 = m + ab$, for some $m \in \mathfrak{m}$ and $b \in R$.

Hence $1 - ab \in \mathfrak{m}$ and $1 - ab \notin R^\times$ $\square$

<u>Lemma</u> (Nakayama) Given a finitely generated R-module M, and an ideal $\mathfrak{a} \subseteq R$ s.t. $\mathfrak{a} \subseteq j(R)$ and $\underbrace{\mathfrak{a} M = M}_{\star}$, then $M = 0$.

Proof

Suppose $M \neq 0$. There is a finite subset $S := \{x_1, \dots, x_n\} \subseteq M$ s.t. $M = \langle S \rangle_R$ and we may assume that it is minimal. By $(\star)$ we have

$$x_n = a_1 x_1 + \dots + a_n x_n,$$

for some $a_1, \dots, a_n \in \mathfrak{a}$. Therefore

$$(1-a_n) x_n = a_1 x_1 + \dots + a_{n-1} x_{n-1}$$

So by the above lemma, $1 - a_n \in R^\times$, thus $\exists\ a_1', \dots, a_{n-1}' \in \mathfrak{a}$ s.t.

$$x_n = a_1' x_1 + \dots + a_{n-1}' x_{n-1},$$

which contradicts the minimality of S $\square$

<u>Corollary 1</u> If M is a finitely generated R-module and $\mathfrak{a} \subseteq j(R)$, and $N \subseteq M$ s.t. $M = N + \mathfrak{a}M$, then $M = N$.

Proof Nakayama's lemma

Clearly $M = N + \mathfrak{a} M \Rightarrow M/N = \mathfrak{a}(M/N) \Rightarrow M/N = 0 \Rightarrow M = N$ $\square$

Corollary 2 Suppose $(R, \mathfrak{m}, k)$ is a local ring and let M be a finitely generated R-module. The quotient module $M/\mathfrak{m}M$ is canonically a k-vector space. Moreover, if $S = \{x_1, \dots, x_n\} \subseteq M$ is s.t. $\langle \tilde{x}_1, \dots, \tilde{x}_n \rangle_k = M/\mathfrak{m}M$, then

$$\langle S \rangle_R = M.$$

Proof

Clearly $M/\mathfrak{m}M = k\tilde{x}_1 + \dots + k\tilde{x}_n \Rightarrow M = Rx_1 + \dots + Rx_n + \mathfrak{m}M$.

So Corollary 2 implies that $M = Rx_1 + \dots + Rx_n$ $\square$

<u>Prop'n</u> Given a Noetherian local ring $(R, \mathfrak{m}, k)$, we have

$$\dim(R) \leqslant \dim_k(\mathfrak{m}/\mathfrak{m}^2) < \infty$$

Proof

As R is Noetherian, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) < \infty$. So $\exists\, a_1, \dots, a_r \in \mathfrak{m}$ s.t.

$$\mathfrak{m}/\mathfrak{m}^2 = (k\,\tilde{a}_1) \oplus \dots \oplus (k\,\tilde{a}_r).$$

Thus by Corollary 2 we have $\mathfrak{m} = \langle a_1, \dots, a_r \rangle_R$. Hence

$$\dim(R) = \underbrace{\mathrm{ht}(\mathfrak{m}) \leqslant r}_{\text{Krull's dim'n th'm}} = \dim_k(\mathfrak{m}/\mathfrak{m}^2) \qquad \square$$

Corollary 3 (Krull's Hauptidealsatz) Let R be a Noetherian ring and let $a \in R \setminus R^\times$ s.t. a is not a zero divisor. Then each minimal prime divisor $\mathfrak{p}$ of $\mathfrak{a} = \langle a \rangle_R$ is s.t. $ht(\mathfrak{p}) = 1$.

Given a Noetherian local ring $(R, \mathfrak{m})$, a *system of local parameters* is a subset $\{x_1, \dots, x_r\} \subseteq \mathfrak{m}$ s.t.

(i) $\dim(R) = r$

(ii) $\operatorname{rad}\langle x_1, \dots, x_r \rangle = \mathfrak{m}$

Corollary 4 If R is Noetherian then $\dim(R[X]) = \dim(R) + 1$.

Proof *

$(\geqslant)$: Let $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_r \in \operatorname{Spec}(R)$. Then we have a strictly ascending chain

$$\mathfrak{p}_0 R[X] \subsetneq \dots \subsetneq \mathfrak{p}_r R[X] \subsetneq \mathfrak{p}_r R[X] + X R[X] \in \operatorname{Spec}(R[X])$$

of length $r+1$. Thus $\dim(R[X]) \geqslant \dim(R) + 1$.

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* pp. 78-79 of Bosch, S., *Algebraic Geometry and Commutative Algebra*, Univeritext, Springer-Verlag.

$(\leq)$: Consider a maximal ideal $\mathfrak{m} \subseteq R[X]$ and define

$$\mathfrak{p} := \mathfrak{m} \cap R \in \operatorname{Spec}(R).$$

<u>Claim</u> We have $\operatorname{ht}(\mathfrak{m}) \leq \operatorname{ht}(\mathfrak{p}) + 1$

Proof of claim

Replacing R by the localisation $R_{\mathfrak{p}}$, we may assume that R is a local ring. Hence $R/\mathfrak{p}$ is a field, so

$$(R/\mathfrak{p})[X] \cong R[X]/\mathfrak{p}R[X]$$

is a PID. Therefore $\exists f \in \mathfrak{m}$ s.t. $\mathfrak{m} = \mathfrak{p}R[X] + fR[X]$. So if

$x_1, \dots, x_r$ is a system of parameters of R, where $r = \dim(R)$, then

$$I := \langle x_1, \dots, x_r, f \rangle_{R[X]}$$

is s.t. $\mathrm{rad}(I) = \mathfrak{m}$. Thus by Krull's dimension theorem,

$$\mathrm{ht}(\mathfrak{m}) \leqslant r+1 = \mathrm{ht}(\mathfrak{p}) + 1.$$

So the claim and thus the lemma follows $\square$