

Lecture 12 Key theorems on transcendence bases

Theorem 4 Let $A, B \subseteq E$ be finite subsets. If

- A is AI

- $A < B$

then $|A| \leq |B|$.

Proof

Write $A = \{\alpha_1, \dots, \alpha_m\}$, $B = \{\beta_1, \dots, \beta_n\}$ and let

$k := |A \cap B|$. If $k = m$ then $A \subseteq B$ and thus $m \leq n$.

So we may assume that $k < m$ and write

$$B = \{\alpha_1, \dots, \alpha_k, \beta_{k+1}, \dots, \beta_m\}$$


Since $\alpha_{k+1} < B$ and $\alpha_{k+1} \notin A$, then there is some β_j s.t.

$$\alpha_{k+1} < \{\alpha_1, \dots, \alpha_k, \beta_{k+1}, \dots, \beta_{j-1}, \beta_j\}$$

but

$$\alpha_{k+1} \notin \{\alpha_1, \dots, \alpha_k, \beta_{k+1}, \dots, \beta_{j-1}\}.$$

So by Lemma 2 (Exchange Property)

$$\beta_j < \{\alpha_1, \dots, \alpha_k, \beta_{k+1}, \dots, \beta_{j-1}, \alpha_{k+1}\}$$

Therefore

$$B < B_1 := B \cup \{\alpha_{k+1}\} \setminus \{\beta_j\}$$

But $A < B$, so by Lemma 3 (Transitivity) $A < B_1$. Hence we may iterate the above argument and thus get a finite sequence of sets

$$B_1, B_2, \dots, B_N$$

s.t.

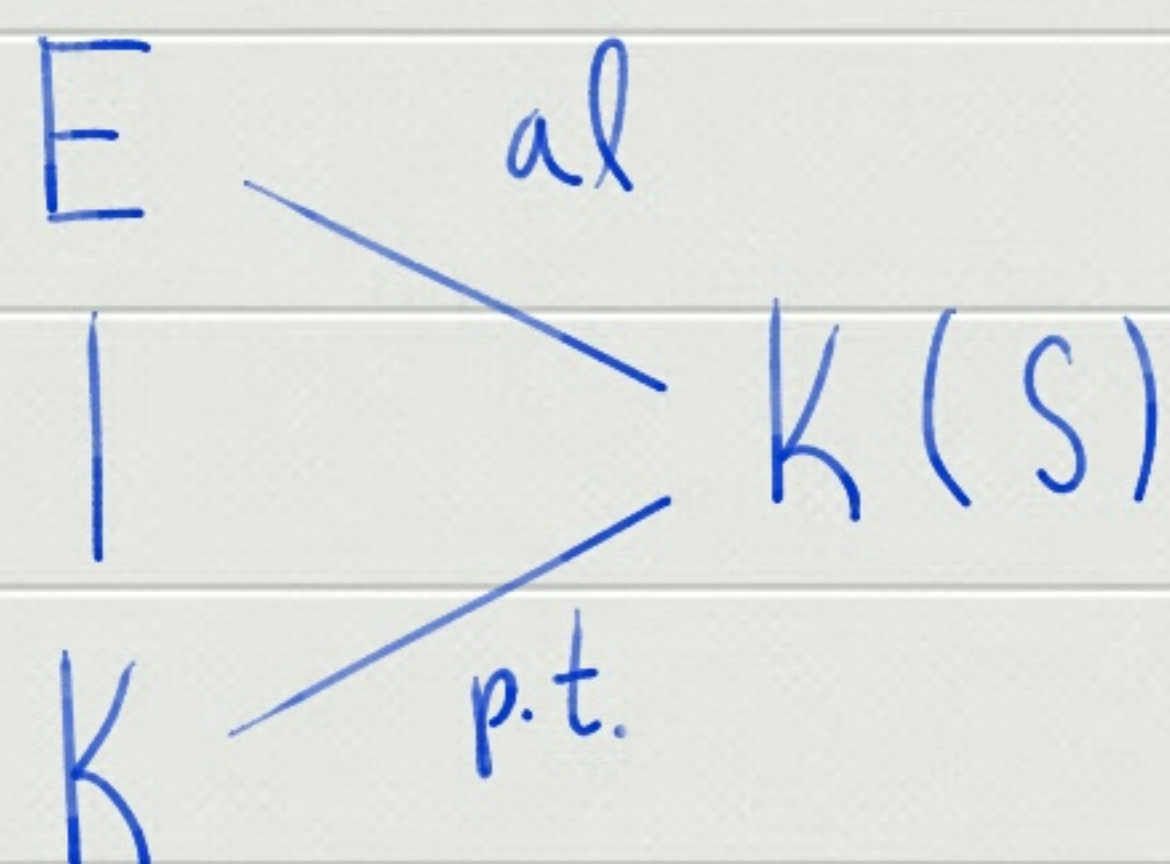
$$|B_r| \leq |B|,$$

for each $r \in \{1, 2, \dots, N\}$, and

$$A \subseteq B_N.$$

Therefore $|A| \leq |B|$. \square

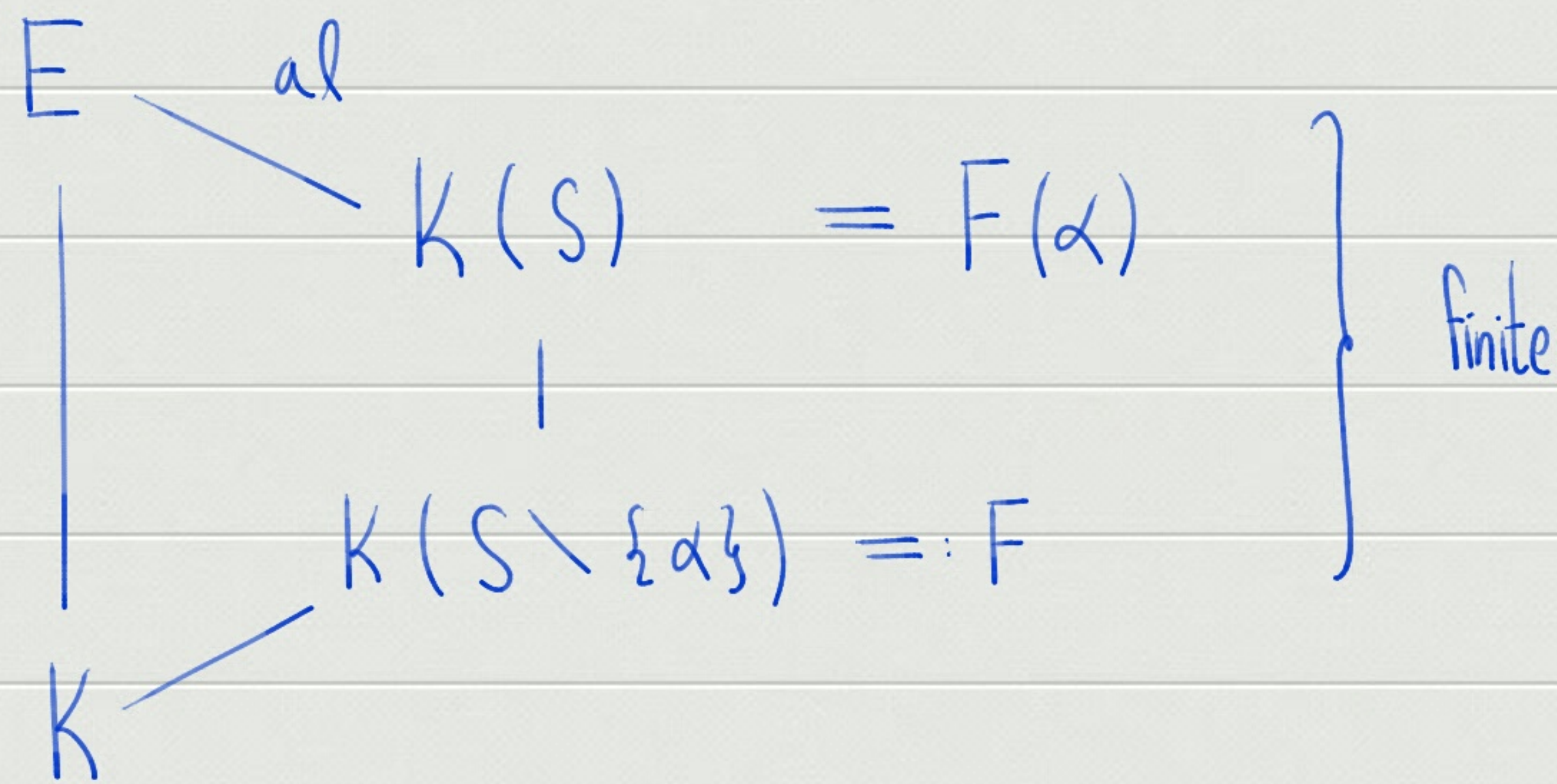
Given any field extension E/K , a transcendence basis is an AI set S such that E is algebraic over $K(S)$, so we have the Hasse diagram



Lemma 5 If E is algebraic over $K(S)$ and S is minimal among the sets with this property, then S is a transcendence basis for E/K .

Proof

Suppose S is not AI. So there is $\alpha \in S$ s.t. $\alpha \notin S \setminus \{\alpha\}$ and thus



By Lemma 3 (i.e. transitivity) it follows that E/F is algebraic, contradicting the minimality condition on S . \square

Theorem 6 If there is a finite $S \subseteq E$ s.t. E is algebraic over $K(S)$, then \exists finite transcendence basis S_1 for E/K .
Moreover, if S_2 another TB for E/K , then $|S_2| = |S_1|$.

Proof

Clearly any minimal $S_1 \subseteq S$ s.t. E is algebraic over $K(S_1)$ is a TB for E/K and Thm 4 implies that

$$|S_2| = |S_1|,$$

where S_2 is any TB for E/K \square