Lecture 12 Key theorems on transcendence bases

Theorem 4 Let $A, B \subseteq E$ be finite subsets. If

- A is AI
- $A \prec B$

then $|A| \leqslant |B|$.

Proof

Write $A = \{\alpha_1, \ldots, \alpha_m\}$, $B = \{\beta_1, \ldots, \beta_n\}$ and let $k := |A \cap B|$. If $k = m$ then $A \subseteq B$ and thus $m \leqslant n$.

So we may assume that $k < m$ and write

$$B = \{\alpha_1, \ldots, \alpha_k, \beta_{k+1}, \ldots, \beta_n\}$$

Since $\alpha_{k+1} < B$ and $\alpha_{k+1} \not< A$, then there is some β_j s.t.

$$\alpha_{k+1} < \{\alpha_1, \ldots, \alpha_k, \beta_{k+1}, \ldots, \beta_{j-1}, \beta_j\}$$

but

$$\alpha_{k+1} \not< \{\alpha_1, \ldots, \alpha_k, \beta_{k+1}, \ldots, \beta_{j-1}\}.$$

So by Lemma 2 (Exchange Property)

$$\beta_j < \{\alpha_1, \ldots, \alpha_k, \beta_{k+1}, \ldots, \beta_{j-1}, \alpha_{k+1}\}$$

Therefore

$$B \prec B_1 := B \cup \{\alpha_{k+1}\} \setminus \{\beta_j\}$$

But $A \prec B$, so by Lemma 3 (Transitivity) $A \prec B_1$. Hence we may iterate the above argument and thus get a finte sequence of sets

$$B_1, B_2, \ldots, B_N$$

s.t.

$$|B_r| \leq |B|,$$

for each $r \in \{1, 2, \ldots, N\}$, and

$$A \subseteq B_N.$$

Therefore $|A| \leq |B|$ $\square$

Given any field extension E/K, a *trascendence basis* is an AI set S such that E is algebraic over $K(S)$, so we have the Hasse diagram

$$\begin{tikzcd}[column sep=small]
E \arrow[dd, no head] \arrow[dr, no head, "\text{al}"] & \\
& K(S) \\
K \arrow[ur, no head, "\text{p.t.}"'] &
\end{tikzcd}$$

<u>Lemma 5</u> If E is algebraic over $K(S)$ and S is minimal among the sets with this property, then S is a trascendence basis for E/K.

Proof

Suppose S is not AI. So there is $\alpha \in S$ s.t. $\alpha \prec S \setminus \{\alpha\}$ and thus

$$\begin{tikzcd}
E \arrow[dr, dash, "\text{al}"] \arrow[ddd, dash] & \\
& K(S) = F(\alpha) \arrow[d, dash, "\text{finite}"] \\
& K(S \setminus \{\alpha\}) =: F \arrow[dl, dash] \\
K &
\end{tikzcd}$$

By Lemma 3 (i.e. transitivity) it follows that E/F is algebraic, contradicting the minimality condition on S $\square$

<u>Theorem 6</u> If there is a finite $S \subseteq E$ s.t. E is algebraic over $K(S)$, then $\exists$ finite transcendence basis S_1 for E/K. Moreover, if S_2 another TB for E/K, then $|S_2| = |S_1|$.

Proof

Clearly any minimal $S_1 \subseteq S$ s.t. E is algebraic over $K(S_1)$ is a TB for E/K and Thm 4 implies that

$$|S_2| = |S_1|,$$

where S_2 is any TB for E/K $\square$