Lecture 13 Basic properties of integral extensions

Let A and B be commutative rings. Suppose that $B \subseteq A$. We say that this ring ext'n is *integral* if $\forall\, x \in A$

$$x^d + b_{d-1}x^{d-1} + \dots + b_1 x + b_0 = 0,$$

for some $b_{d-1}, \dots, b_0 \in B$. We say that $B \subseteq A$ is *finite* if A regarded as a B-module is finetely generated, i.e. $A = \langle S \rangle_B$, for a finite $S \subseteq A$.

Given a ring R and a (left) R-module M, the annihilator of a subset $S \subseteq M$ is

$$\mathrm{Ann}_R(S) := \{ r \in R \mid \forall s \in S : rs = 0 \}.$$

In particular, if $S = M$ then $\mathrm{Ann}_R(M) = \ker(\rho)$, where

$$\begin{aligned} \rho : R &\longrightarrow \mathrm{End}(M) \\ r &\longmapsto \begin{pmatrix} M \longrightarrow M \\ x \longmapsto r \cdot x \end{pmatrix} \end{aligned}$$

is the repn attached to M over R.

<u>Prop'n 1</u> Suppose that $B \subseteq A$ is an ext'n of rings and let $x \in A$.
The following are equivalent

(1) x is integral over B

(2) $B[x]$ is finite over B

(3) $\exists$ inter. ring $B \subseteq A' \subseteq A$ containing x and finite over B

(4) $\exists$ subring $A' \subseteq A$ s.t $xA' \subseteq A'$ & A' is finite over B

(5) $\exists$ faithfull $B[x]$-module M that is finite over B

Proof

$(1) \Rightarrow (2)$: We have $x^d + b_d x^{d-1} + \dots + b_1 x + b_0 = 0$, so

$$x^d = -b_d x^{d-1} - \dots - b_1 x - b_0$$
$$x^{d+1} = -b_d x^d - \dots - b_1 x^2 - b_0 x$$
$$\vdots$$

which may be used to express each $x^d, x^{d+1}, \dots$ as a B-linear combination of $S := \{1, x, x^2, \dots, x^{d-1}\}$, so $\langle S \rangle_B = B[x]$ is finite over B. Thus (2) follows.

(2) $\Rightarrow$ (3): Put $A' := B[x]$ and (3) follows.

(3) $\Rightarrow$ (4): Clearly the inter. ring $B \subseteq A' \subseteq A$ containing x of (3) is s.t.

$$xA' \subseteq A', \qquad \text{☆}$$

so (4) follows.

(4) $\Rightarrow$ (5): Let $M := A'$ from (4) and note that (☆) implies M is also a $B[x]$-module. As $1 \in A'$, we have

$$\mathrm{Ann}_{B[x]}(M) = 0,$$

so (5) follows.

(5) $\Rightarrow$ (1): We have a f.g. B-module $M = \langle m_1, \dots, m_n \rangle$, so $\exists \ (r_{ij}) \in M_n(B)$ s.t. $\forall i \in \{1, \dots, n\}$:

$$x m_i = \sum_{j=1}^{n} r_{ij} m_j .$$

Put $(x \delta_{ij} - r_{ij}) =: (\alpha_{ij}) \in M_n(A)$ so that $\forall i \in \{1, \dots, n\}$:

$$\sum_{j=1}^{n} \alpha_{ij} m_j = 0$$

and

$$\det(\alpha_{ij}) \, m_i = 0.$$

But $\det(\alpha_{ij}) \in B[x]$ and also

$$\det(\alpha_{ij}) \in \mathrm{Ann}_{B[x]}(B) = 0.$$

Hence $\det(\alpha_{ij}) = 0$ and thus, by expanding $\det(\alpha_{ij})$, we get the integral eq'n

$$x^d + b_{d-1} x^{d-1} + \dots + b_1 x + b_0 = 0,$$

where $b_0, \dots, b_{d-1} \in B$ and (1) follows $\square$

Lemma Consider a ring ext'n $B \subseteq A$. If $x_1, \ldots, x_n \in A$ are integral over B then $B[x_1, \ldots, x_n]$ is finite over B.

Proof

Clear $\square$

Integral dependence is transitive, as shown as follows.

Prop'n 2 Given a chain $C \subseteq B \subseteq A$ of commutative rings:

C integral over B & B integral over A $\Longrightarrow$ C integral over A

Proof

Suppose $x \in A$ is integral over B, so $\exists\ b_0, \dots, b_{d-1} \in B$ s.t.

$$x^d + b_{d-1} x^{d-1} + \dots + b_1 x + b_0 = 0.$$

By the above lemma the ring

$$B' := C[b_0, \dots, b_{d-1}]$$

is finite over C. Moreover, $B'[x]$ is finite over B'. Thus have a Hasse diagram

$$
\begin{tikzcd}
B[x] \arrow[dd, dash, "\text{fin.}"'] \arrow[dr, dash] & \\
& B'[x] \arrow[dd, dash, "\text{fin.}"'] \arrow[ddd, bend left, "\text{fin.}"] \\
B \arrow[dr, dash] & \\
& B' \arrow[d, dash, "\text{fin.}"'] \\
& C
\end{tikzcd}
$$

As $x \in B'[x]$, Prop'n 1 implies that x is integral over C and the prop'n follows $\square$