Lecture 14 Dimension of integral extensions

Thm (Noether) For each finitely generated K-algebra $K[\alpha_1, \dots, \alpha_n] := A$ $\exists$ AI set $\{t_1, \dots, t_d\} \subseteq A$ over K s.t. A is integral over R:

$$
\begin{tikzcd}
A \arrow[dd, dash] \arrow[dr, dash, "\text{int}"] & \\
& K[t_1, \dots, t_d] =: R \\
K \arrow[ur, dash, "\text{p.t.}"'] &
\end{tikzcd}
$$

Proof — by induction on n following Nagata

Case $n = 0$: Clear.

Induction step:

If $\{\alpha_1, \dots, \alpha_n\}$ AI the thm is clear. So we may assume that, say, α_n depends algebraically on $\{\alpha_1, \dots, \alpha_{n-1}\}$, i.e. $\exists\ G(X) \in K[\alpha_1, \dots, \alpha_{n-1}, X] =: A$ s.t.

$$G(X) \neq 0,$$

$$G(\alpha_n) \neq 0.$$

Clearly $\forall\, \mu \in \mathbb{Z}_{>0}^{n-1}$ $\exists!$ automorphism $f_\mu \in \mathrm{Aut}(\mathcal{A})$ defined on the set $\{\alpha_1, \dots, \alpha_{n-1}, X\}$ of generators of the K-algebra $\mathcal{A}$ by

$$\begin{aligned} \alpha_i &\longmapsto \alpha_i + X^{\mu_i} \qquad (\forall\, i \in \{1, \dots, n-1\}) \\ X &\longmapsto X \end{aligned}$$

Claim There is $\mu \in \mathbb{Z}_{>0}^{n-1}$ s.t. $f_\mu(G)$ has leading term

$$G^* := f_\mu(G) = a_d X^d + \dots$$

with $a_d \in K^\times$.

Proof of claim

Any $G \in \mathcal{A}$ we have $G = \sum_I a_{\nu_1 \cdots \nu_n} \alpha_1^{\nu_1} \cdot \ldots \cdot \alpha_{n-1}^{\nu_{n-1}} X^{\nu_n}$, so

$$f_\mu(G) = \sum_I a_{\nu_1 \cdots \nu_n} (\alpha_1 + X^{\mu_1})^{\nu_1} \cdot \ldots \cdot (\alpha_{n-1} + X^{\mu_{n-1}})^{\nu_{n-1}} X^{\nu_n}$$

$$= a_{\nu_1 \cdots \nu_n} X^{\nu_n + \mu_1 \nu_1 + \ldots + \mu_{n-1} \nu_{n-1}} + \ldots$$

If r is any integer $>$ all the ν's s.t. $a_{\nu_1 \cdots \nu_n} \neq 0$, then $\forall\, i \in \{1, \ldots, n-1\}$ we put $\mu_i := r^i$. The claim follows with

$$d := \max \{ \nu_n + \mu_1 \nu_1 + \ldots + \mu_{n-1} \nu_{n-1} \mid a_{\nu_1 \cdots \nu_n} \neq 0 \}$$

$\square$

Let μ be as in the claim and define $\forall\, i \in \{1, \dots n-1\}$

$$z_i := f_\mu^{-1}(\alpha_i).$$

Note that

$$g^*(z_1, \dots, z_{n-1}\ \alpha_n) = 0.$$

Thus α_n is integrally dependent on $K[z_1, \dots, z_{n-1}]$, and so is

$$\alpha_1 = z_1 + \alpha_n^{\mu_1}, \ \dots, \ \alpha_{n-1} = z_{n-1} + \alpha_n^{\mu_{n-1}}.$$

Hence <u>A is integral over $K[z_1, \dots, z_{n-1}] =: A'$</u>. Then, as the latter K-algebra is generated by $n-1$ elements, the inductive hypothesis implies that $\exists$ AI set $\{t_1, \ldots, t_d\} \subseteq A'$ over K s.t. A' is integral over $K[t_1, \ldots, t_d] =: R$, thus the Hasse diagram

$$\begin{tikzcd}
A \arrow[d, dash, "\text{int}", "*"'] \arrow[ddr, dash, bend left] & \\
A' \arrow[dd, dash] \arrow[dr, dash, "\text{int}"] & \\
& R \arrow[dl, dash, "\text{p.t.}"] \\
K &
\end{tikzcd}$$

int $\Leftarrow$ Prop'n 2 (i.e. transitivity) $\square$

<u>Lemma 2</u> Suppose that A/B is an integral extn where A is a domain.

$$B \text{ is a field} \iff A \text{ is a field}$$

Proof

$(\Longrightarrow)$: For each non-zero $y \in A$: $\exists$ monic $f(X) \in B[X]$ s.t.

$$f(y) = 0$$

We may assume WLOG that $f(X)$ has $\overbrace{\text{minimal}}^{*}$ degree. We have

$$y^n + b_{n-1} y^{n-1} + \dots + b_1 y + b_0 = 0 \qquad (\star)$$

If $b_0 = 0$, then $y(y^{n-1} + \dots + b_1) = 0$. But A is an integral domain, so $y^{n-1} + \dots + b_1 = 0$, which contradicts the minimality condition $(*)$. Thus $b_0 \neq 0$ and $(\star)$ yields in the field of fractions of A (where A embeds)

$$y^{n-1} + b_{n-1} y^{n-2} + \dots + b_1 + \frac{b_0}{y} = 0,$$

$$-\frac{1}{b_0}\left(y^{n-1} + b_{n-1} y^{n-2} + \dots + b_1\right) = \frac{1}{y} \in A.$$

($\Leftarrow$): For each non-zero $x \in B$ we have $\frac{1}{x} \in A$, so there are $b_{n-1}, \dots, b_0 \in B$ s.t.

$$\left(\frac{1}{x}\right)^n + b_{n-1}\left(\frac{1}{x}\right)^{n-1} + \dots + b_1\left(\frac{1}{x}\right) + b_0 = 0$$

Multiplying it by x^{n-1} we get

$$\frac{1}{x} + b_{n-1} + \dots + b_0 x^{n-1} = 0$$

$$-\left(b_{n-1} + \dots + b_0 x^{n-1}\right) = \frac{1}{x} \in B \qquad \square$$

Given a commutative ring R, let

$$\mathrm{Spec}(R) := \{ \mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal of } R \}$$

Each ring homomorphism $\varphi : B \longrightarrow A$ gives a map

$$\begin{aligned} \mathrm{Spec}(A) &\xrightarrow{\varphi^*} \mathrm{Spec}(B) \\ \mathfrak{q} &\longmapsto \varphi^{-1}(\mathfrak{q}) \end{aligned}$$

<u>Remark</u> Later we'll provide $\mathrm{Spec}(\cdot)$ with a topology known as the Zariski topology, where φ^* is continuous.

If φ is the inclusion map, then

$$\varphi^*(\mathfrak{q}) = \mathfrak{q} \cap B$$

and say that the point $\mathfrak{q}$ is above $\mathfrak{p} = \mathfrak{q} \cap B$ and express this with the following diagram

$$\begin{array}{ccc} \mathfrak{q} & \qquad & \mathrm{Spec}(A) \\ | & & \downarrow \\ \mathfrak{p} & & \mathrm{Spec}(B) \end{array}$$

We'll think of $\mathfrak{q}$ as lying on the fibre above the point $\mathfrak{p}$.

Prop'n 3 As before, let A / B be an integral ring ext'n. Then

(a) $\forall\, \mathfrak{p} \in \operatorname{Spec}(B) : \exists$

$$\begin{array}{ccc} \mathfrak{q} & \quad & \operatorname{Spec}(A) \\ | & & \downarrow \\ \mathfrak{p} & & \operatorname{Spec}(B) \end{array}$$

(b)

$$\begin{array}{ccc} \mathfrak{q} \subseteq \mathfrak{q}' & \quad & \operatorname{Spec}(A) \\ \diagdown \;\; \diagup & & \downarrow \\ \mathfrak{p} & & \operatorname{Spec}(B) \end{array} \quad \Longrightarrow \quad \mathfrak{q} = \mathfrak{q}'$$

(c)

$$\begin{array}{ccc} \mathfrak{q} & \quad & \operatorname{Spec}(A) \\ | & & \downarrow \\ \mathfrak{p} & & \operatorname{Spec}(B) \end{array} \quad \Longrightarrow \quad \left[\begin{array}{c} \mathfrak{q} \text{ maximal} \\ \Updownarrow \\ \mathfrak{p} \text{ maximal} \end{array} \right]$$

Proof

(c): $\begin{array}{c} A/\mathfrak{q} \\ | \text{ int.} \\ B/\mathfrak{p} \end{array}$ & $\underline{\text{Lemma 2}}$ $\Longrightarrow$ $\left[\begin{array}{c} \mathfrak{q} \text{ maximal} \\ \Updownarrow \\ \mathfrak{p} \text{ maximal} \end{array}\right]$.

(b): If $S \subseteq B$ is a multiplicative set, then $\forall\, x \in A$ & $s \in S$

$$x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0 = 0$$

$$\frac{x^n}{s^n} + \frac{b_{n-1}}{s} \cdot \frac{x^{n-1}}{s^{n-1}} + \dots + \frac{b_1}{s^{n-1}} \cdot \frac{x}{s} + \frac{b_0}{s^n} = 0$$

Therefore

$$\begin{array}{c} AS^{-1} \\ \Big| \;\; \text{int} \\ BS^{-1} \end{array}$$

In particular,

$$\begin{array}{rl} A_p & := AS_p^{-1} \\ \Big| \;\; \text{int} & \\ B_p & := BS_p^{-1} \end{array},$$

where $S_p := B \setminus p$. Hence

$$
\begin{array}{ccc}
\mathfrak{q} \subseteq \mathfrak{q}' & & \\
\diagdown \quad \diagup & & \\
\mathfrak{p} & &
\end{array}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, dash] & A_{\mathfrak{p}} \arrow[d, dash, "int"] \\
B \arrow[r] & B_{\mathfrak{p}}
\end{tikzcd}
\qquad
\begin{array}{c}
\mathfrak{q} S_{\mathfrak{p}}^{-1} =: \mathfrak{n} \subseteq \mathfrak{n}' := \mathfrak{q}' S_{\mathfrak{p}}^{-1} \\
\diagdown \quad \diagup \\
\mathfrak{m}
\end{array}
$$

By (c) both $\mathfrak{n}$ & $\mathfrak{n}'$ are maximal. But $\mathfrak{n} \subseteq \mathfrak{n}'$, thus $\mathfrak{n} = \mathfrak{n}'$.

It is easy to see that there is a 1-1 correspondence

$$
\begin{aligned}
\{ \mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \cap S = \emptyset \} &\xrightarrow{\sim} \operatorname{Spec}(RS^{-1}) \\
\mathfrak{p} &\longmapsto \mathfrak{p} S^{-1}
\end{aligned}
$$

Therefore $\mathfrak{q} = \mathfrak{q}'$ $\square$

(a): Pick a maximal ideal $\mathfrak{n} \subseteq A_{\mathfrak{p}}$. Then (c) implies that $\mathfrak{n} \cap B_{\mathfrak{p}}$ is maximal and thus $\mathfrak{n} \cap B_{\mathfrak{p}} =: \mathfrak{m}$ is the unique maximal ideal of the local ring $B_{\mathfrak{p}}$. Hence

$$
\begin{tikzcd}
\mathfrak{q} := \kappa_2^{-1}(\mathfrak{n}) \arrow[d, dash] & A \arrow[r, "\kappa_2"] \arrow[d, dash] & A_{\mathfrak{p}} \arrow[d, dash, "\mathrm{int}"] & \mathfrak{n} \arrow[d, dash] \\
\mathfrak{p} = \kappa_1^{-1}(\mathfrak{m}) & B \arrow[r, "\kappa_1"] & B_{\mathfrak{p}} & \mathfrak{m}
\end{tikzcd}
$$

The prop'n follows $\square$

Th'm 2 (Cohen - Seidenberg 1^{st})

(i) If

$$\begin{array}{ccccccccc} \mathfrak{q}_0 & \subsetneq & \mathfrak{q}_1 & \subsetneq & \cdots & \subsetneq & \mathfrak{q}_r & \in & \operatorname{Spec}(A) \\ | & & | & & & & | & & \downarrow \\ \Rightarrow \mathfrak{p}_0 & \subsetneq & \mathfrak{p}_1 & \subsetneq & \cdots & \subsetneq & \mathfrak{p}_r & \in & \operatorname{Spec}(B) \end{array}$$

(ii) If $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r \in \operatorname{Spec}(B)$ & if $\exists\ \mathfrak{q}_0 \in \operatorname{Spec}(A)$ s.t. $\mathfrak{q}_0$ is above $\mathfrak{p}_0$, then there are

$$\begin{array}{ccccccccc} \mathfrak{q}_0 & \subsetneq & \mathfrak{q}_1 & \subsetneq & \cdots & \subsetneq & \mathfrak{q}_r & \in & \operatorname{Spec}(A) \\ | & & | & & & & | & & \downarrow \\ \mathfrak{p}_0 & \subsetneq & \mathfrak{p}_1 & \subsetneq & \cdots & \subsetneq & \mathfrak{p}_r & \in & \operatorname{Spec}(B) \end{array}$$

Proof

(i): Follows from (b) of Prop'n 3.

(ii): By induction on r. The case $r=0$ is clear. Now assume that it is true for $r-1$. From (a) of Prop'n 3 we have

$$
\begin{tikzcd}[row sep=small]
\cdots \subsetneq \mathfrak{q}_{r-1} \arrow[d, dash] & \subsetneq & \mathfrak{q}_r \arrow[d, dash] := \pi_2^{-1}(\tilde{\mathfrak{q}}_r) & & A \arrow[r, "\pi_2"] \arrow[d, dash] & A/\mathfrak{q}_{r-1} \arrow[d, dash, "int"] & \exists & \tilde{\mathfrak{q}}_r \arrow[d, dash] \\
\cdots \subsetneq \mathfrak{p}_{r-1} & \subsetneq & \mathfrak{p}_r := \pi_1^{-1}(\tilde{\mathfrak{p}}_r) & & B \arrow[r, "\pi_1"] & B/\mathfrak{p}_{r-1} & & \tilde{\mathfrak{p}}_r
\end{tikzcd}
$$

$\square$

Corollary If A is an integral ext'n of B, then their respective Krull dimensions coinside, $\dim(A) = \dim(B)$.

Proof

By the Cohen-Seidenberg 1^{st} th'm each chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_r \in \operatorname{Spec}(B) \qquad \bigstar'$$

lifts to a chain

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \dots \subsetneq \mathfrak{q}_r \in \operatorname{Spec}(A). \qquad \bigstar$$

Conversely, by Prop'n 3 each chain $(\bigstar)$ restricts to a chain $(\bigstar')$ $\square$

<u>Th'm</u> If A finitely generated K-algebra and A is an integral domain, then

$$\dim(A) = \operatorname{tr\,deg}_K(A)$$

Proof

By Noether's Normalisation Lemma $\exists$ an algebraically independent $\{t_1, \dots, t_d\} \subseteq A$ over K s.t.

$$\begin{tikzcd}
A \arrow[dr, dash, "\text{int}"] \arrow[dd, dash] & \\
& K[t_1, \dots, t_d] =: R \\
K \arrow[ur, dash, "\text{p.t.}"'] &
\end{tikzcd}$$

By the corollary to the 1st Cohen - Seidenberg th'm we have

$$\dim(A) = \dim\left(K[t_1, \ldots, t_d]\right)$$

But the Corollary 4 of Krull's dimension theorem implies that

$$\dim\left(K[t_1, \ldots, t_d]\right) = d$$

and the theorem follows $\square$