Lecture 16 Some consequences of the Nullstellensatz

From Hilbert's Nullstellensatz we have $1-1$ correspondences:

$$\begin{array}{ccc} \mathbb{A}^n & \xrightarrow{\sim} & \{\text{maximal ideals } \mathfrak{m} \subseteq \overline{K}[X_1, \dots, X_n]\} \\ \cup & & \cup \\ V & \xrightarrow{\sim} & \{\text{maximal ideals } \mathfrak{m} \subseteq \overline{K}[V]\} \end{array}$$

$$(x_1, \dots, x_n) \longmapsto \langle X_1 - x_1, \dots, X_n - x_n \rangle$$

$$\begin{array}{ccc} \{\text{irreducible subsets of } \mathbb{A}^n\} & \xrightarrow{\sim} & \operatorname{Spec}(\overline{K}[X_1, \ldots, X_n]) \\ \uparrow & & \uparrow \\ \{\text{irreducible subsets of } V\} & \xrightarrow{\sim} & \operatorname{Spec}(\overline{K}[V]) \\ W & \longmapsto & \mathcal{I}(W) \end{array}$$

Here V is a fixed algebraic set. Recall that $\mathbb{A}^n$ is equipped with the Zariski topology, where $\mathcal{U} \subseteq \mathbb{A}^n$ is open if

$$\mathcal{U} = \mathbb{A}^n \setminus \mathcal{V}(I),$$

for some ideal $I \subseteq \overline{K}[X_1, \ldots, X_n]$. We say that a closed set V is *irreducible* if

$$V = V_1 \cup V_2,$$

with closed $V_i \subseteq V$, implies either $V_1 = V$ or $V_2 = V$.