Lecture 17 Preliminaries

Given any A-module M we define

$$\mathrm{Ass}_A(M) := \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid \exists\, v \in M \text{ s.t. } \mathfrak{p} = \mathrm{Ann}(v) \},$$

where

$$\mathrm{Ann}(v) := \{ a \in A \mid av = 0 \}$$

In other words, $\mathrm{Ass}_A(M)$ may be naturally identified with the set of cyclic submodules $Av \subseteq M$, $v \in M$, s.t. $Av \cong A/\mathfrak{p}$ with $\mathfrak{p} \in \mathrm{Spec}(A)$, i.e.

$$0 \longrightarrow \mathfrak{p} \longrightarrow A \longrightarrow Av \longrightarrow 0$$
$$a \longmapsto a \cdot v$$

is exact.

Prop'n 1.1 Notation as above, assume that A is Noetherian. The following are equivalent

(i) $M = 0$

(ii) $\mathrm{Ass}_A(M) = \emptyset$

Proof

(i) $\Rightarrow$ (ii) : Clear.

$\neg$(i) $\Rightarrow$ $\neg$(ii) : We have the following.

Claim If the set

$$\mathcal{S} := \{\mathrm{Ann}(v) \subseteq A \mid v \in M \ \&\ v \neq 0\}$$

has a maximal element $\mathfrak{p}_{\mathcal{S}}$, then $\mathfrak{p}_{\mathcal{S}} \in \mathrm{Ass}_A(M)$.

Proof of claim

Suppose there is nonzero $v \in M$ s.t. $\mathfrak{p}_S = \mathrm{Ann}(v)$. If a and $b \in A$ are s.t.

$$ab \in \mathfrak{p}_S \qquad \& \qquad b \notin \mathfrak{p}_S$$

then

$$a \cdot (b \cdot v) = (ab) \cdot v = 0 \quad \& \quad b \cdot v \neq 0.$$

So $\mathfrak{p}_S \subseteq \langle \mathfrak{p}_S, a \rangle_A \subseteq \mathrm{Ann}(b \cdot v)$. Hence the maximality of $\mathfrak{p}_S$ in S we have

$$\mathfrak{p}_S = \langle \mathfrak{p}_S, a \rangle_A,$$

i.e. $a \in \mathfrak{p}_S$. Thus $\mathfrak{p}_S \in \mathrm{Spec}(A)$ and the claim follows.

If $M \neq 0$ and A is Noetherian, then such $\mathfrak{p}_S$ exists and the prop'n follows $\square$

Prop'n 1.2 Let A be a Noetherian ring and consider a nonzero, finitely generated A-module M. Then there exist $\mathfrak{p}_1, \ldots, \mathfrak{p}_m \in \operatorname{Spec}(A)$ & a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M$$

s.t.

$$\begin{array}{ccc} M_1/M_0 & & M_m/M_{m-1} \\ \cong & \cdots & \cong \\ A/\mathfrak{p}_1 & & A/\mathfrak{p}_m \end{array}$$

Proof

As A is Noetherian and $M \neq 0$, Prop'n 1.1 says that there is $\mathfrak{p}_1 \in \operatorname{Ass}(M)$, i.e. there is a submodule $M_1 \subseteq M$ s.t

$$M_1 \cong A/\mathfrak{p}_1.$$

If $M_1 \subsetneq M$, then again by Prop'n 1.1, there is $\mathfrak{p}_2 \in \operatorname{Ass}(M/M_1)$ and thus

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subseteq M$$

s.t.

$$\begin{array}{ccc} M_1/M_0 & M_2/M_1 & \\ \cong & \cong & \cdots \\ A/\mathfrak{p}_1 & A/\mathfrak{p}_2 & \end{array}$$

The ACC says that this process terminates $\square$

The *length* $\ell(M)$ of an A-module M is the length ℓ of the longest possible chain of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{\ell-1} \subsetneq M_\ell = M$$

We say that a family of A-submodules $\{M_\alpha\}_{\alpha \in \Lambda}$ is *directed* if $\forall \alpha, \beta \in \Lambda \; \exists \gamma \in \Lambda : M_\alpha \subseteq M_\gamma \; \& \; M_\beta \subseteq M_\gamma$.

<u>Lemma 1.1</u> Let M be an A-module s.t. $M = \bigcup_{\alpha \in \Lambda} M_\alpha$ where $\{M_\alpha\}_{\alpha \in \Lambda}$ is directed, then $\ell_A(M) = \sup_\alpha (\ell_A(M_\alpha))$.

Proof

Clear $\square$

<u>Lemma 1.2</u> If A is a domain, but not a field, then $\ell_A(A) = \infty$.

Proof

There are nonzero ideal $J \subsetneq A$ and $0 \neq a \in J$. So $\cdots \subsetneq a^2 A \subsetneq aA \subsetneq A$, as

$$a^{k+1}A = a^k A \Rightarrow a^{k+1} r = a^k \Rightarrow ar = 1 \Rightarrow J = A,$$

which contradicts $J \subsetneq A$. Hence $\ell_A(A) = \infty$ $\square$

Recall that for any multiplicative subset $S \subseteq A$ of a ring A, and an A-module M, the *localisation*

$$S^{-1}M := (M \times S)/\sim,$$

where $(v, s) \sim (v', s') \iff \exists\, t \in S$ s.t. $t(vs' - v's) = 0$. We write

$$\frac{v}{s} := [(v, s)]_{\sim}$$

so

$$\ker\begin{pmatrix} M \longrightarrow S^{-1}M \\ v \longmapsto \frac{v}{1} \end{pmatrix} = \{ v \in M \mid \exists\, \alpha \in S \text{ s.t. } \alpha v = 0 \}.$$

We also have

$$S^{-1}M \cong M \otimes_A (S^{-1}A).$$

So if A is a domain, the *rank* of M is

$$\operatorname{rnk}(M) := \dim_K (M \otimes_A K) = \max\{ |S| \mid \text{linearly independent } S \subseteq M \}$$

where K is the field of fractions of A. Recall that given R-modules M_1, M_2, N, and an R-bilinear map $b : M_1 \times M_2 \longrightarrow N$ there is a unique R-linear map $\hat{b}$ s.t.

$$
\begin{tikzcd}
x_1 \otimes x_2 \arrow[rrr, mapsto, bend left] & M_1 \otimes M_2 \arrow[r, "\hat{b}"] & N & b(x_1, x_2) \\
& M_1 \times M_2 \arrow[u, "\kappa"] \arrow[ur, "b"'] & & \\
& (x_1, x_2) \arrow[uul, mapsto] \arrow[uurr, mapsto] & &
\end{tikzcd}
$$

<u>Lemma 1.3</u> For each A-module M and each ideal $\mathfrak{a} \subseteq A$ we have an isomorphism

$$\begin{aligned} (A/\mathfrak{a}) \otimes_A M &\longrightarrow M/\mathfrak{a}M \\ \bar{a} \otimes v := (a+\mathfrak{a}) \otimes v &\longmapsto av + \mathfrak{a}M =: \overline{av} \end{aligned} \qquad \star$$

Proof

The composition

$$\begin{tikzcd} A \times M \arrow[r] \arrow[rr, bend left] & M \arrow[r] & M/\mathfrak{a}M \end{tikzcd}$$

$$(a, v) \mapsto av \longmapsto \overline{av}$$

induces an A-bilinear map $(A/\mathfrak{a}) \times M \xrightarrow{b} M/\mathfrak{a}M$ and $\hat{b}$ is the A-linear map ($\star$) with inverse

$$\begin{aligned} M/\mathfrak{a}M &\xrightarrow{f} (A/\mathfrak{a}) \otimes_A M . \\ \bar{v} &\longmapsto \bar{1}_A \otimes v \end{aligned}$$

Indeed,

$$\begin{array}{ccccc} (A/\mathfrak{a}) \otimes M & \xrightarrow{\hat{b}} & M/\mathfrak{a}M & \xrightarrow{f} & (A/\mathfrak{a}) \otimes M \\ \bar{a} \otimes v & \longmapsto & \overline{av} & \longmapsto & \bar{1}_A \otimes (av) = \bar{a} \otimes v \end{array}$$

and

$$\begin{array}{ccccc} M/\mathfrak{a}M & \xrightarrow{f} & (A/\mathfrak{a}) \otimes M & \xrightarrow{\hat{b}} & M/\mathfrak{a}M \\ \bar{v} & \longmapsto & \bar{1}_A \otimes v & \longmapsto & \overline{1_A v} = \bar{v} \end{array}$$

$\square$

<u>Lemma 1.4</u> Let A be a Noetherian domain s.t. $\dim(A) = 1$ and M an A-module with $M_{tors} = 0$ and finite $\operatorname{rnk}(M) =: r$. Then for each nonzero $a \in A$ we have

$$\ell_A(M/aM) \leq r \cdot \ell_A(A/aA) < \infty$$

Proof

<u>Case 1</u> M is finitely generated. Choose a linearly independent subset $\mathcal{B} \subseteq M$ s.t. $|\mathcal{B}| = r$. Put $E := \langle \mathcal{B} \rangle_A$ and $C := M/E$.

<u>Claim</u> There is nonzero $t \in A$ s.t. $tC = 0$.

Proof of claim

We have $M = \langle v_1, \dots, v_g \rangle_A$. But $M = S^{-1}E$, where $S = A \setminus \{0\}$. Thus

$$v_k \in t_k^{-1} \cdot E,$$

where $t_k \in S$, for each $k \in \{1, \dots, g\}$. So $t := \prod_{i=1}^{g} t_g$ does the job.

By Prop'n 1.2 there exist $\mathfrak{p}_1, \dots, \mathfrak{p}_m \in \mathrm{Spec}(A)$ & a chain of submodules

$$0 = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{m-1} \subseteq C_m = C \qquad \star$$

$$\begin{array}{ccc} C_1 / C_0 & & C_m / C_{m-1} \\ \cong & \cdots & \cong \\ A/\mathfrak{p}_1 & & A/\mathfrak{p}_m \end{array} \qquad \text{(all over } A\text{)}$$

By the above claim we have for each $i \in \{1, \dots, m\}$, $t \in \mathfrak{p}_i$ thus $\mathfrak{p}_i \neq 0$.
But $\dim(A) = 1$. Therefore $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ are maximal and $(\star)$ is composition series.
By the Jordan-Hölder theorem

$$\ell_A(C) = m < \infty . \qquad *$$

For each $n \in \mathbb{Z}_{\geqslant 1}$, tensoring the short exact sequence

$$0 \to E \to M \to C \to 0$$

by the quotient module $A/\mathfrak{a}$, where $\mathfrak{a} := a^n A$, gives the right exact sequence

$$(A/\mathfrak{a}) \otimes_A E \to (A/\mathfrak{a}) \otimes_A M \to (A/\mathfrak{a}) \otimes_A C \to 0$$

and Lemma 1.3 gives a further right exact sequence

$$E/a^n E \to M/a^n M \to C/a^n C \to 0$$

Therefore

$$\ell_A(M/a^n M) \leqslant \ell_A(E/a^n E) + \ell_A(C/a^n C) \qquad \dagger$$

<u>Claim</u> For each $n \in \mathbb{Z}_{\geqslant 1}$

$$\ell_A(M/a^n M) = n \cdot \ell_A(M/aM) \tag{1}$$

$$\ell_A(E/a^n E) = n \cdot r \cdot \ell_A(A/aA) \tag{2}$$

Proof of claim

As $M_{tors} = 0$, for each $k \in \mathbb{Z}_{\geqslant 0}$ we have an A-module isomorphism

$$\varphi_k : a^k M \xrightarrow{\sim} a^{k+1} M$$
$$a^k v \longmapsto a^{k+1} v$$

Therefore

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathfrak{a}^{k+2}M \arrow[r] \arrow[u] & \mathfrak{a}^{k+1}M \arrow[r] \arrow[u] & \mathfrak{a}^{k+1}M/\mathfrak{a}^{k+2}M \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathfrak{a}^{k+1}M \arrow[r] \arrow[u] & \mathfrak{a}^{k}M \arrow[r] \arrow[u] & \mathfrak{a}^{k}M/\mathfrak{a}^{k+1}M \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

where the 2 rows and the 3 columns are exact. Hence each successive quotient of

$$\mathfrak{a}^n M \subseteq \mathfrak{a}^{n-1}M \subseteq \cdots \subseteq \mathfrak{a}^2 M \subseteq \mathfrak{a}M \subseteq M$$

is isomorphic to $M/\mathfrak{a}M$. The same thus holds for the chain

$$0 \subseteq \mathfrak{a}^{n-1}M / \mathfrak{a}^n M \subseteq \cdots \subseteq \mathfrak{a}M / \mathfrak{a}^n M \subseteq M / \mathfrak{a}^n M$$

and (1) follows. Similarily,

$$\ell_A(E/\mathfrak{a}^n E) = n \cdot \ell_A(E/\mathfrak{a}E).$$

But $E \cong A^r$, so

$$E/\mathfrak{a}E \cong (A/\mathfrak{a}A)^r$$

and thus

$$\ell_A(E/\mathfrak{a}E) = r \cdot \ell_A(A/\mathfrak{a}A)$$

Hence (2) and the claim follows.

This claim together with the inequality $(\dagger)$ gives

$$n\, l_A(M/aM) \leqslant rn\, l_A(A/aA) + l_A(C/a^n C)$$

and thus

$$l_A(M/aM) \leqslant r \cdot l_A(A/aA) + \frac{1}{n} \cdot l_A(C/a^n C).$$

But from $(*)$ we have

$$l_A(C/a^n C) \leqslant l_A(C) < \infty$$

So Case 1 follows by taking n large enough.

Case 2 General M. Consider the family $\{T_\alpha\}_{\alpha \in \Lambda}$, where $T_\alpha := M_\alpha / (M_\alpha \cap aM)$ and $M_\alpha \subseteq M$ is a finitely generated submodule, so

$$M_\alpha / aM_\alpha \longrightarrow T_\alpha \longrightarrow 0$$

is a right exact sequence and thus

$$\ell_A(T_\alpha) \leqslant \ell_A(M_\alpha / aM_\alpha) \overset{C1}{\leqslant} r \cdot \ell_A(A/aA),$$

for each $\alpha \in \Lambda$. But $\{T_\alpha\}_{\alpha \in \Lambda}$ is a directed family of A-modules s.t.

$$M/aM = \bigcup_{\alpha \in \Lambda} T_\alpha .$$

By Lemma 1.1 $\ell_A(M/aM) \leqslant r \cdot \ell_A(A/aA)$ and the lemma follows $\square$