Lecture 18 The Krull-Akizuki theorem

Theorem 1.1 (Krull-Akizuki) Let A be a 1-dim'l Noetherian domain with field of fractions K, let L/K be a field ext'n s.t. $[L:K] < \infty$ and B a ring s.t.

$$\begin{tikzcd} & L \\ B \arrow[ur, dash] & \\ & K \arrow[uu, dash] \\ A \arrow[uu, dash] \arrow[ur, dash] & \end{tikzcd}$$

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Then

(1) $\ell_A(B/J) < \infty$, for each nonzero ideal $J \subseteq B$

(2) B is Noetherian & $\dim(B) \leqslant 1$

Proof

(1): WLOG we may assume that L is the field of fractions of B so that B is a torsion free A-module of rank $r := [L:K]$. Consider a nonzero ideal $J \subseteq B$.

<u>Claim</u> There is a nonzero element $a \in J \cap A$.

Proof of claim

There is nonzero $b \in J$ and we may pick $a_0, \dots, a_m \in A$ s.t.

$$a_m b^m + \dots + a_1 b + a_0 = 0$$

with m of minimal degree. But $B \subseteq L$, so B is a domain and thus $\overset{\in J \cap A}{a_0} \neq 0$. The claim follows.

Therefore

$$\ell_A(B/J) \leq \overbrace{\ell_A(B/aB) \leq r \cdot \ell_A(A/aA) < \infty}^{L1.4},$$

where the first inequality comes from the right exact sequence $B/aB \longrightarrow B/J \longrightarrow 0$ and (1) follows.

(2): We have the left exact sequence $0 \to J/aB \to B/aB$, hence

$$\ell_B(J/aB) \leq \ell_A(J/aB) \leq \overbrace{\ell_A(B/aB) < \infty}^{L1.4}.$$

Therefore $\ell_B(J/aB) < \infty$, so J/aB is a Noetherian B-module.
But we have a short exact sequence of B-modules

$$0 \longrightarrow aB \longrightarrow J \longrightarrow J/aB \longrightarrow 0$$

Also, as A is a domain and $a \neq 0$ we have a B-module isomorphism

$$B \xrightarrow{\sim} aB$$
$$b \longmapsto ab$$

So aB is a Noetherian B-module as well. Thus J is Noetherian B-module.
Finally, if $\dim(B) > 1$ then $\exists\ \mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(B)$ s.t.

$$0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq B.$$

Thus the domain $B/\mathfrak{p}_1$ is not a field, so by Lemma 1.2

$$\ell_B(B/\mathfrak{p}_1) = \infty,$$

which contradicts (1). Hence $\dim(B) \leq 1$ and (2) follows $\square$

We say that A is a *Dedekind* ring if A is Noetherian, integrally closed, and s.t. $\dim(A) = 1$.

<u>Corollary 1</u> If in ($\star$) we further assume that A is integrally closed (so that A is Dedekind) and that B is the integral closure of A in L, then B is Dedekind.

<u>Remark</u> Suppose that L/K is purely inseparable. So a suitable power of Frobenius, $\varphi(x) = x^{p^r}$ (where $p = \chi(K) > 0$ and suitable $r \in \mathbb{Z}_{\geqslant 1}$) gives

$$\begin{tikzcd}
\mathcal{O}_{R,L} \arrow[d, bend right=60, "\varphi"'] & & \mathrm{Spec}(\mathcal{O}_{R,L}) \arrow[d, "\iota^*"] \\
R \arrow[u, "\iota"'] \arrow[r, phantom, "\longmapsto"] & & \mathrm{Spec}(R) \arrow[u, bend left=60, "\varphi^*"]
\end{tikzcd}$$

As $\mathcal{O}_{R,L}/R$ is integral, ι^* is a surjective morphism. Moreover, ι^* is in fact an isomorphism, i.e.

$$\varphi^* \circ \iota^* = 1_{\mathrm{Spec}(\mathcal{O}_{R,L})}$$

Indeed, for all $\mathfrak{P} \in \mathrm{Spec}(\mathcal{O}_{R,L})$ we have the equality

$$\varphi^{-1}(\mathfrak{P} \cap R) = \mathfrak{P}.$$

$(\subseteq)$: If $x \in \varphi^{-1}(\mathfrak{P} \cap R)$ then $x^{p^r} \in \mathfrak{P} \cap R$. So $x \in \mathfrak{P}$ as $\mathfrak{P}$ is prime.

$(\supseteq)$: If $x \in \mathfrak{P}$ then $x^{p^r} \in \mathfrak{P}$, but $x^{p^r} \in R$, so $x \in \varphi^{-1}(\mathfrak{P} \cap R)$.

<u>Prop'n 1.3</u> Let R be a Noetherian domain of dimension one. For each nonzero ideal $\mathfrak{a} \subseteq R$ we have a unique product decomposition into primary ideals

$$\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_g$$

where $\mathrm{rad}(\mathfrak{q}_1), \ldots, \mathrm{rad}(\mathfrak{q}_g) \in \mathrm{Spec}(R)$ are distinct.

Proof

As we assumed that R is Noetherian, we have the minimal primary decomposition

$$\mathfrak{a} = \bigcap_i \mathfrak{q}_i, \qquad *$$

where $\mathfrak{p}_i := \mathrm{rad}(\mathfrak{q}_i) \in \mathrm{Spec}(R)$, for each $i \in \{1, \ldots, g\}$. But R is a domain with Krull dimension $\dim(R) = 1$. Thus $\mathfrak{p}_1, \ldots, \mathfrak{p}_g \in \operatorname{Spec}(R)$ are distinct maximal ideals and thus for each $i, j \in \{1, \ldots, g\}$ s.t. $i \neq j$ we have

$$\mathfrak{p}_i + \mathfrak{p}_j = R,$$

so

$$\mathfrak{q}_i + \mathfrak{q}_j = R.$$

<u>Claim</u> More generally, let $I_1, \ldots, I_n \subseteq R$ be ideals s.t. $j \neq i$ implies that $I_i + I_j = R$, for all $i, j \in \{1, \ldots, n\}$. Then

$$\bigcap_{i=1}^{n} I = \prod_{i=1}^{n} I_i$$

Proof of claim — by induction on n

Case $n = 2$: So we have

$$x_1 + x_2 = 1,$$

where $x_1 \in I_1$ and $x_2 \in I_2$. Then $\forall\, y \in I_1 \cap I_2$

$$y x_1 + y x_2 = y.$$

But $y x_1 \in I_1 I_2$ and $y x_2 \in I_1 I_2$, hence

$$I_1 \cap I_2 \subseteq I_1 I_2.$$

The reverse inclusion is clear, so the equality follows.

Case $n > 2$: $I'_{n-1} := I_{n-1} \cap I_n$. So by Case 1, $I'_{n-1} := I_{n-1} I_n$.

Now we apply the inductive hypothesis to the $n-1$ ideals $I_1, \ldots, I_{n-2}, I'_{n-1}$,

$$\bigcap_{i=1}^{n} I_i = I_1 \cap \cdots \cap I_{n-2} \cap I'_{n-1} = I_1 \cdots I_{n-2} I'_{n-1} = \prod_{i=1}^{n} I_i$$

and the claim follows.

The prop'n follows if we apply the claim to $(\ast)$ $\square$