Lecture 19 Basics of primary ideals

Prop'n 2.1 If A is a Noetherian ring and $\mathfrak{a} \subseteq A$ an ideal, then there is $n \in \mathbb{Z}_{\geqslant 1}$ s.t.

$$\operatorname{rad}(\mathfrak{a})^n \subseteq \mathfrak{a}$$

Proof

As A is Noetherian $\exists\ x_1, \dots, x_k \in A$ s.t. $\operatorname{rad}(\mathfrak{a}) = \langle x_1, \dots, x_k \rangle_A$.

There are $n_1, \dots, n_k \in \mathbb{Z}_{\geqslant 1}$ s.t. $x_i^{n_i} \in \mathfrak{a}$, $\forall\, i \in \{1, \dots, k\}$

and we may define

$$m := 1 + \sum_{i=1}^{k} (n_i - 1)$$

so that

$$\operatorname{rad}(\mathfrak{a})^m = \langle x_1^{r_1} \cdots x_k^{r_k} \mid r_1 + \cdots + r_k = m \rangle_A .$$

For each partition

$$r_1 + \cdots + r_k = m$$

there is an index i s.t. $r_i \geq n_i$. Thus each monomial $x_1^{r_1} \cdots x_k^{r_k} \in \mathfrak{a}$.

Therefore $\text{rad}(\mathfrak{a})^m \subseteq \mathfrak{a}$ $\square$

We say that an ideal $\mathfrak{a} \subseteq R$ is *primary* if every zero divisor of $R/\mathfrak{a}$ is nilpotent.

<u>Prop'n 2.2</u> Given a Noetherian ring R and ideals $\mathfrak{q}, \mathfrak{m} \subseteq R$ with $\mathfrak{m}$ maximal, the following are eq't

(i) $\mathfrak{q}$ is $\mathfrak{m}$-primary

(ii) $\text{rad}(\mathfrak{q}) = \mathfrak{m}$

(iii) There is $n \in \mathbb{Z}_{\geq 1}$ s.t. $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$

Proof

(i $\Rightarrow$ ii): Directly from the definition of an $\mathfrak{m}$-primary ideal $\mathfrak{q}$.

(ii $\Rightarrow$ i): Note that $\mathrm{rad}(\mathfrak{q}) = \mathfrak{m}$ implies that the image $\overline{\mathfrak{m}}$ of $\mathfrak{m}$ with respect to

$$R \longrightarrow R/\mathfrak{q}$$

consists of the nilpotent elements of $R/\mathfrak{q}$. So it suffices to show the following.

<u>Lemma 2.1</u> Given a commutative ring R and an ideal $\mathfrak{a} \subseteq R$, we have the implication

$$\mathrm{rad}(\mathfrak{a}) =: \mathfrak{m} \text{ is a maximal ideal} \Rightarrow (R/\mathfrak{a}, \overline{\mathfrak{m}}) \text{ is a local ring.}$$

Proof of lemma

WLOG we may assume $\mathfrak{a} = 0$. The nilradical $\mathfrak{N}_R = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$. So for each $\mathfrak{p} \in \mathrm{Spec}(R)$ we have $\mathfrak{N}_R \subseteq \mathfrak{p}$. Thus for all maximal ideals $\mathfrak{m} \subseteq R$

$$\mathfrak{N}_R \subseteq \mathfrak{m}. \qquad \star$$

If the ideal $\mathfrak{N}_R$ is maximal, then each inclusion $(\star)$ is an equality, so $(R, \mathfrak{N}_R)$ is a local ring and the lemma follows.

Indeed, as $(R/\mathfrak{q}, \overline{\mathfrak{m}})$ is local, $(R/\mathfrak{q})^\times = (R/\mathfrak{q}) \setminus \overline{\mathfrak{m}}$ and (i) follows.

(ii $\Rightarrow$ iii): It follows from Prop$\underline{n}$ 2.1.

(iii $\Rightarrow$ ii): $\mathfrak{m} = \mathrm{rad}(\mathfrak{m}^n) \subseteq \mathrm{rad}(\mathfrak{q}) \subseteq \mathrm{rad}(\mathfrak{m}) = \mathfrak{m}$ $\square$