Lecture 20 Discrete valuation rings

We say that a commutative ring R is a *local ring* if R has a unique maximal ideal $\mathfrak{m} \subseteq R$. If a local ring R is s.t.

(i) R is a PID (so in particular R is integrally closed)

(ii) $|\operatorname{Spec}(R)| = 2$ (so that R is not a field)

then we say that R is a *discrete valuation ring* and call any element $t \in \mathfrak{m}$ that generates $\mathfrak{m} = \langle t \rangle_R$ a *local parameter*. Note that for each nonzero x in the field of fractions F of R we have a unique element $\operatorname{ord}(x) \in \mathbb{Z}$ s.t.

$$x = u\, t^{\nu(x)}, \quad \text{where } u \in R^\times.$$

Prop'n 2.3 Let $(R, \mathfrak{m}, k)$ be a Noetherian local domain s.t. $\dim(R) = 1$. The following are equivalent

(i) R is a DVR

(ii) R is integrally closed

(iii) The maximal ideal $\mathfrak{m}$ is principal

(iv) The Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ has dimension one (over k).

(v) For each nonzero ideal $\mathfrak{a} \subseteq R$ there is $n \in \mathbb{Z}$ s.t. $\mathfrak{a} = \mathfrak{m}^n$.

(vi) There is $t \in R$ s.t. for each nonzero ideal $\mathfrak{a} \subseteq R$ s.t. $\mathfrak{a} = t^n R$

Proof

(i) $\Rightarrow$ (ii): We shall use the following.

<u>Lemma 2.2</u> Suppose B is a *valuation* ring of a field K, that is, B is an integral domain with field of fractions K s.t for each $x \in F^{\times}$ either $x \in B$ or $x^{-1} \in B$. Then B is integrally closed.

Proof of Lemma

Let $x \in K$ be integral over B, so $\exists\ b_0, \ldots, b_{n-1} \in B$ s.t.

$$x^n + b_{n-1}x^{n-1} + \cdots + b_0 = 0 \qquad *$$

Case 1. $x \in B$: Then we are done.

Case 2. $x \in B$: Then $x^{-1} \in B$, so if we divide (*) by x^{1-n} we have

$$x + b_{n-1} x^{-1} + \cdots + b_0 x^{1-n} = 0$$

thus

$$x = -\left(b_{n-1} x^{-1} + \cdots + b_0 x^{1-n} \right) \in B$$

and the lemma follows.

As a DVR is a valuation ring, (ii) follows.

(ii) $\Rightarrow$ (iii): We shall use the following.

Lemma 2.3 If $\mathfrak{a} \subsetneq R$ is a nonzero ideal of a $\underbrace{\text{1-dimensional local domain } (R, \mathfrak{m})}_{*}$ then there is $n \in \mathbb{Z}_{\geq 1}$ s.t.

$$\mathfrak{m}^n \subseteq \mathfrak{a} \ \& \ \mathfrak{m}^{n-1} \nsubseteq \mathfrak{a}.$$

Proof of lemma

Each $\mathfrak{a}$ as above is $\mathfrak{m}$-primary. Indeed, $(*)$ implies that $\mathfrak{p} = \mathfrak{m}$ is the only prime ideal such that $\mathfrak{a} \subseteq \mathfrak{p}$, hence

$$\operatorname{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R) \text{ s.t. } \mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p} = \mathfrak{m}.$$

So by Prop'n 2.2 $\exists n \in \mathbb{Z}_{\geq 1}$ s.t. $\mathfrak{m}^n \subseteq \mathfrak{a}$ & $\mathfrak{m}^{n-1} \nsubseteq \mathfrak{a}$. The lemma follows.

Pick a nonzero $a \in \mathfrak{m}$. By Lemma 2.3 $\exists n \in \mathbb{Z}_{\geq 1}$ s.t. $\mathfrak{m}^n \subseteq Ra$ & $\mathfrak{m}^{n-1} \nsubseteq Ra$.

So there is $b \in \mathfrak{m}^{n-1} \setminus Ra$. Let F be the field of fractions of R and put

$$x := \frac{a}{b} \in F$$

As $b \notin Ra$, then $x^{-1} \notin R$. But $\overbrace{R \text{ is integrally closed}}^{(ii)}$. Thus x^{-1} is not integral over R, so

$$x^{-1}\mathfrak{m} \nsubseteq \mathfrak{m} \qquad (R \text{ is Noetherian}) \qquad \star$$

<u>Claim</u> We have $x^{-1}\mathfrak{m} \subseteq R$, an ideal of R.

Proof of claim

Since $b \in \mathfrak{m}^{n-1}$, we have $\forall\, \alpha \in \mathfrak{m} : b\alpha \in \mathfrak{m}^n \subseteq Ra$. Thus $x^{-1}\mathfrak{m} \subseteq R$.

Clearly $x^{-1}\mathfrak{m}$ is an ideal and the claim follows.

But $(R, \mathfrak{m}, k)$ is a local ring, so $R \setminus \mathfrak{m} = R^\times$ and $(\star)$ implies that

$$x^{-1}\mathfrak{m} = R.$$

Thus

$$\mathfrak{m} = xR$$

and (iii) follows.

Variant of Nakayama's Lemma

(iii) $\Rightarrow$ (iv): Given a finitely generated module M over a local ring $(R, \mathfrak{m}, k)$ and $x_1, \dots, x_n \in M$ s.t. $\{\bar{x}_1, \dots, \bar{x}_n\} \subseteq M/\mathfrak{m}M$ is a k-basis, we have $M = \langle x_1, \dots, x_n \rangle_R$. In particular, for $M = \mathfrak{m}$ we have $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq 1$ as $\mathfrak{m} = \langle x_1 \rangle_R$ by (iii). To obtain equality we'll use the following.

Lemma 2.4 If $(R, \mathfrak{m})$ is a Noetherian local ring, then exactly one of the following holds true:

(a) $\mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \mathfrak{m}^3 \supsetneq \cdots$

(b) $\exists\, n \in \mathbb{Z}_{\geq 1}$ s.t $\mathfrak{m}^n = 0$. In which case R is Artinian.

Proof of lemma

Suppose $\neg$ (a), so $\exists\, n \in \mathbb{Z}_{\geq 1}$ s.t. $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. Recall that Nakayama's lemma says that for every f.g. R-module M and $\mathfrak{a} \subseteq J(R)$: $\mathfrak{a}M = M \Rightarrow M = 0$. Here $J(R)$ is Jacobson's radical. So if $M = \mathfrak{m}^n$ & $\mathfrak{a} = \mathfrak{m}$, then $\mathfrak{m}^n = 0$.

So for each $\mathfrak{p} \in \operatorname{Spec}(R)$ we have

$$\mathfrak{m}^n \subseteq \mathfrak{p}$$

Taking radicals we get $\mathfrak{m} \subseteq \mathfrak{p}$, thus $\mathfrak{m} = \mathfrak{p}$ i.e. (b) and the lemma follows.

As $\dim(R) = 1$, the ring R is not Artinian. So by Lemma 2.4 $\mathfrak{m}/\mathfrak{m}^2 \neq 0$. But we have shown that

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leqslant 1.$$

Therefore this inequality is in fact an equality and (iv) follows.

(iv) $\Rightarrow$ (v): As $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$, there is $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. In particular, $t \neq 0$, and Lemma 2.4 implies that we have a chain of ideals

$$\cdots \subsetneq \mathfrak{m}^3 \subsetneq \mathfrak{m}^2 \subsetneq \mathfrak{m}$$

Indeed, (b) implies that $t^n = 0$, contradicting R is an integral domain, so (a) follows.

Claim For each $i \in \mathbb{Z}_{\geq 1}$ we have $\dim_k(\mathfrak{m}^i / \mathfrak{m}^{i+1}) = 1$.

Proof of claim

As $\mathfrak{m}/\mathfrak{m}^2 = \langle \bar{t} \rangle_k$ then the variant of Nakama's lemma implies $\mathfrak{m} = \langle t \rangle_R$ and

$$k = R/\mathfrak{m} \xrightarrow{\sim} \mathfrak{m}^i / \mathfrak{m}^{i+1}$$

$$\bar{x} \longmapsto t^i \bar{x}$$

Indeed, as $\mathfrak{m}^n = t^n R = \langle t^n \rangle_R$ then $\mathfrak{m}^n / (\mathfrak{m}\, \mathfrak{m}^n) = \langle \overline{t^n} \rangle_k$. So the claim follows.

By Lemma 2.3, for each nonzero ideal $\mathfrak{a} \subsetneq R$ there is $n \in \mathbb{Z}_{\geq 1}$ s.t

$$\mathfrak{a} \subseteq \mathfrak{m}^n \ \& \ \mathfrak{a} \nsubseteq \mathfrak{m}^{n+1}.$$

So there is $a \in \mathfrak{a} \setminus \mathfrak{m}^{n+1}$ and the above claim implies that

$$\mathfrak{m}^n / \mathfrak{m}^{n+1} = \langle \bar{a} \rangle_k$$

Therefore the variant of Nakayama's Lemma yields

$$\mathfrak{m}^n = \langle a \rangle_R = Ra \subseteq \mathfrak{a}$$

Thus $\mathfrak{a} = \mathfrak{m}^n$ and (v) follows.

(v) $\Rightarrow$ (vi): As R is not Artinian, Lemma 2.4 implies that $\mathfrak{m} \neq \mathfrak{m}^2$, so there is $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. But $Rt = \mathfrak{m}^n$, for some $n \in \mathbb{Z}_{\geq 1}$, thus $n = 1$ and $Rt = \mathfrak{m}$, so $Rt^n = \mathfrak{m}^n$. The claim follows.

(vi) $\Rightarrow$ (i): We have $Rt = \mathfrak{m}$, so for each $k \in \mathbb{Z}_{\geq 1}$,

$$Rt^k \supsetneq Rt^{k+1}$$

So $\forall$ nonzero $a \in R$ $\exists!$ k_0 s.t. $Ra = Rt^{k_0}$; define $\nu(a) := k_0$ $\square$

<u>Corollary</u> If R is a Dedekind domain, then for each nonzero ideal $\mathfrak{a} \subseteq R$ there exist unique $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_g\} \subseteq \operatorname{Spec}(R)$ s.t.

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}.$$