Lecture 21 Completion of a discrete valuation ring

Let $(R, \mathfrak{m})$ be a discrete valuation ring and let $t \in \mathfrak{m}$ be a local parameter.

We have a group homomorphism

$$F^{\times} \longrightarrow \mathbb{Z}$$

$$x \longmapsto v(x)$$

and extend it to a map

$$F \longrightarrow \mathbb{Z} \cup \{\infty\}$$

by letting $v(0) = \infty$. This map is known as the *valuation map* and it is such that for all $x, y \in F$:

$$v(x+y) \leqslant \min\{v(x), v(y)\}.$$

So if we fix any $c \in \mathbb{R}_{>1}$, then we get a map

$$|\cdot|_v : F \longrightarrow \mathbb{R}_{\geqslant 0}$$

by letting

$$x \longmapsto |x|_v := \begin{cases} c^{-v(x)} & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$$

This map is an absolute value map as it is s.t.

(A1) $\forall \; x \in F$: $\qquad |x|_v = 0 \iff x = 0$

(A2) $\forall \; x, y \in F$: $\qquad |xy|_v = |x|_v |y|_v$

(A3) $\forall \; x, y \in F$: $\qquad |x+y|_v \leqslant |x|_v + |y|_v$

In fact it satisfies a property stronger than (A3), namely

(A3') $\forall\, x, y \in F$: $\quad |x+y|_v \leqslant \max\{|x|_v, |y|_v\}$.

So we may say that it is a *nonarchimedean* absolute value. We may recover the local ring structure $(R, \mathfrak{m})$ from the absolute value. Indeed, we have

$$R = \{x \in F \mid |x|_v \leqslant 1\},$$

$$\mathfrak{m} = \{x \in F \mid |x|_v < 1\}.$$

All DVR's may be obtained from nonarchimedean absolute values.

As $|\cdot|_\nu$ is an absolute value, the map $d_\nu(x,y) := |x-y|_\nu$ turns F into a metric space. Indeed, we have

(M1) $\forall\, x, y \in F : \; d_\nu(x,y) = 0 \iff x = y,$

(M2) $\forall\, x, y \in F : \; d_\nu(x,y) = d_\nu(y,x),$

(M3) $\forall\, x, y, z \in F : \; d_\nu(x,z) \leqslant d_\nu(x,y) + d_\nu(y,z).$

Recall that a metric space (M, d) is *complete* if every Cauchy sequence $\{x_i\}_{i=1}^{\infty}$ there is $x \in M$ s.t.

$$\lim_{i \to \infty} x_i = x$$

The completion $\overline{M}$ of M is an isometry $\eta : M \longrightarrow \overline{M}$ that satisfies the following universal property. Given any isometry $f : M \longrightarrow C$, where C is complete, there is a unique isometry $\hat{f} : \overline{M} \longrightarrow C$ that makes the diagram

$$\begin{tikzcd} \overline{M} \arrow[r, "\hat{f}"] & C \\ M \arrow[u, "\kappa"] \arrow[ur, "f"'] & \end{tikzcd}$$

commute. If $M = F$ and $d(\cdot,\cdot)$ comes from an absolute value, then

$$\overline{F} = \mathcal{C}_F / \mathcal{N}_F,$$

where $\mathcal{C}_F$ is the ring of Cauchy sequences $\{x_i\}_{i=1}^{\infty}$ with ring structure given component-wise, and $\mathfrak{m}_F \subseteq \mathcal{C}_F$ is the maximal ideal that consists of the sequences $\{x_i\}_{i=1}^{\infty}$ s.t.

$$\lim_{i \to \infty} x_i = 0.$$