Lecture 1 Preliminaries

Given any A-module M we define

$$\mathrm{Ass}_A(M) := \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid \exists\, v \in M \text{ s.t. } \mathfrak{p} = \mathrm{Ann}(v) \},$$

where

$$\mathrm{Ann}(v) := \{ a \in A \mid av = 0 \}$$

In other words, $\mathrm{Ass}_A(M)$ may be naturally identified with the set of cyclic submodules $Av \subseteq M$, $v \in M$, s.t. $Av \cong A/\mathfrak{p}$ with $\mathfrak{p} \in \mathrm{Spec}(A)$, i.e.

$$0 \longrightarrow \mathfrak{p} \longrightarrow A \longrightarrow Av \longrightarrow 0$$
$$a \longmapsto a \cdot v$$

is exact.

Prop'n 1.1 Notation as above, assume that A is Noetherian. The following are equivalent

(i) $M = 0$

(ii) $\mathrm{Ass}_A(M) = \phi$

Proof

(i) $\Rightarrow$ (ii) : Clear.

$\neg$(i) $\Rightarrow \neg$(ii) : We have the following.

Claim If the set

$$\mathcal{S} := \{\mathrm{ann}(v) \subseteq A \mid v \in M \ \& \ v \neq 0\}$$

has a maximal element $\mathfrak{p}_{\mathcal{S}}$, then $\mathfrak{p}_{\mathcal{S}} \in \mathrm{Ass}_A(M)$.

Proof of claim

Suppose there is nonzero $v \in M$ s.t. $\mathfrak{p}_{\mathcal{S}} = \mathrm{Ann}(v)$. If a and $b \in A$ are s.t.

$$ab \in \mathfrak{p}_{\mathcal{S}} \qquad \& \qquad b \notin \mathfrak{p}_{\mathcal{S}}$$

then

$$a \cdot (b \cdot v) = (ab) \cdot v = 0 \quad \& \quad b \cdot v \neq 0.$$

So $\mathfrak{p}_{\mathcal{S}} \subseteq \langle \mathfrak{p}_{\mathcal{S}}, a \rangle_A \subseteq \mathrm{Ann}(b \cdot v)$. Hence the maximality of $\mathfrak{p}_{\mathcal{S}}$ in $\mathcal{S}$ we have

$$\mathfrak{p}_{\mathcal{S}} = \langle \mathfrak{p}_{\mathcal{S}}, a \rangle_A,$$

i.e. $a \in \mathfrak{p}_{\mathcal{S}}$. Thus $\mathfrak{p}_{\mathcal{S}} \in \mathrm{Spec}(A)$ and the claim follows.

If $M \neq 0$ and A is Noetherian, then such $\mathfrak{p}_{\mathcal{S}}$ exists and the prop'n follows $\square$

Prop'n 1.2 Let A be a Noetherian ring and consider a nonzero, finitely generated A-module M. Then there exist $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \operatorname{Spec}(A)$ & a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

s.t.

$$\begin{array}{ccc} M_1/M_0 & & M_n/M_{n-1} \\ \cong & \cdots & \cong \\ A/\mathfrak{p}_1 & & A/\mathfrak{p}_n \end{array}$$

Proof

As A is Noetherian and $M \neq 0$, Prop'n 1.1 says that there is $\mathfrak{p}_1 \in \mathrm{Ass}(M)$, i.e. there is a submodule $M_1 \subseteq M$ s.t

$$M_1 \cong A/\mathfrak{p}_1.$$

If $M_1 \subsetneq M$, then again by Prop'n 1.1, there is $\mathfrak{p}_2 \in \mathrm{Ass}(M/M_1)$ and thus

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subseteq M$$

s.t.

$$\begin{array}{ccc} M_1/M_0 & M_2/M_1 & \\ \cong & \cong & \cdots \\ A/\mathfrak{p}_1 & A/\mathfrak{p}_2 & \end{array}$$

The ACC says that this process terminates $\square$

The *length* $\ell(M)$ of an A-module M is the length ℓ of the longest possible chain of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{\ell-1} \subsetneq M_\ell = M$$

We say that a family of A-submodules $\{M_\alpha\}_{\alpha\in\Lambda}$ is *directed* if $\forall \alpha, \beta \in \Lambda \ \exists \gamma \in \Lambda : M_\alpha \subseteq M_\gamma \ \& \ M_\beta \subseteq M_\gamma$.

<u>Lemma 1.1</u> Let M be an A-module s.t. $M = \bigcup_{\alpha\in\Lambda} M_\alpha$ where $\{M_\alpha\}_{\alpha\in\Lambda}$ is directed, then $\ell_A(M) = \sup_\alpha(\ell_A(M_\alpha))$.

Proof

Clear $\square$

Lemma 1.2 If A is a domain, but not a field, then $\ell_A(A) = \infty$.

Proof

There are nonzero ideal $J \subsetneq A$ and $0 \neq a \in J$. So $\cdots \subsetneq a^2 A \subsetneq aA \subsetneq A$, as

$$a^{k+1}A = a^k A \Rightarrow a^{k+1} r = a^k \Rightarrow ar = 1 \Rightarrow J = A,$$

which contradicts $J \subsetneq A$. Hence $\ell_A(A) = 0$ $\square$

If A is a domain, the *rank* of M is the dimension

$$\operatorname{rnk}(M) := \dim_K (M \otimes_A K)$$

of the K-vector space $M \otimes_A K$, where K is the field of fractions of A. We have

$$\operatorname{rnk}(M) = \max\{|S| \mid \text{linearly independent } S \subseteq M\}, \text{ since}$$

$$M_{tors} = \ker\begin{pmatrix} M \longrightarrow M \otimes_A K \\ v \longmapsto v \otimes 1_A \end{pmatrix}.$$

<u>Lemma 1.3</u> For each A-module M and each ideal $\mathfrak{a} \subseteq A$ we have an isomorphism

$$(A/\mathfrak{a}) \otimes_A M \xrightarrow{\sim} M/\mathfrak{a}M \qquad \star$$

$$\bar{a} \otimes v := (a+\mathfrak{a}) \otimes v \longmapsto av + \mathfrak{a}M =: \overline{av}$$

Proof

The composition

$$\begin{tikzcd} A \times M \arrow[r] \arrow[rr, bend left] & M \arrow[r] & M/\mathfrak{a}M \end{tikzcd}$$

$$(a, v) \mapsto av \longmapsto \overline{av}$$

induces a A-bilinear map $(A/\mathfrak{a}) \times M \longrightarrow M/\mathfrak{a}M$ and thus the A-linear map $(\star)$. The inverse of the latter is $\bar{v} \mapsto \bar{1}_A \otimes v$ $\square$

<u>Lemma 1.4</u> Let A be a Noetherian domain s.t. $\dim(A) = 1$ and M an A-module with $M_{tors} = 0$ and finite $\mathrm{rnk}(M) =: r$. Then for each nonzero $a \in A$ we have

$$\ell_A(M/aM) \leq r \cdot \ell_A(A/aA) < \infty$$

Proof

Case 1 M is finitely generated. Choose a linearly independent subset $\mathcal{B} \subseteq M$ s.t. $|\mathcal{B}| = r$. Put

$$E := \langle \mathcal{B} \rangle_A ,$$

$$C := M / E .$$

Claim There is nonzero $t \in A$ s.t. $tC = 0$.

Proof of claim

Have $M = \langle v_1, \ldots, v_g \rangle_A$ and $t_1 \cdot v_1, \ldots, t_g \cdot v_g \in E$, so $t := \prod_{i=1}^{g} t_g$ does the job.

By Prop'n 1.2 there exist $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \operatorname{Spec}(A)$ & a chain of submodules

$$0 = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{n-1} \subseteq C_n = C \qquad \star$$

$$\begin{array}{ccc} C_1 / C_0 & & C_n / C_{n-1} \\ \cong & \cdots & \cong \\ A/\mathfrak{p}_1 & & A/\mathfrak{p}_n \end{array}$$

By the above claim we have for each $i \in \{1, \dots, n\}$, $t \in \mathfrak{p}_i$ thus $\mathfrak{p}_i \neq 0$.
But $\dim(A) = 1$. Therefore $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are maximal and $(\star)$ is composition series.
By the Jordan-Hölder theorem

$$\ell_A(C) = n < \infty . \qquad *$$

For each $n \in \mathbb{Z}_{\geqslant 1}$, tensoring the short exact sequence

$$0 \to E \to M \to C \to 0$$

by the quotient ring $A/\mathfrak{a}$, where $\mathfrak{a} := a^n A$, gives the right exact sequence

$$(A/\mathfrak{a}) \otimes_A E \to (A/\mathfrak{a}) \otimes_A M \to (A/\mathfrak{a}) \otimes_A C \to 0$$

and Lemma 1.3 gives a further right exact sequence

$$E/a^n E \to M/a^n M \to C/a^n C \to 0$$

Therefore

$$\ell_A(M / a^n M) \leq \ell_A(E / a^n E) + \ell_A(C / a^n C)$$

Claim For each $n \in \mathbb{Z}_{\geq 1}$

$$\ell_A(M / a^n M) = n \cdot \ell_A(M / aM) \qquad (1)$$

$$\ell_A(E / a^n E) = n \cdot r \cdot \ell_A(A / aA) \qquad (2)$$

Proof of claim

As $M_{tors} = 0$, each successive quotient of

$$a^n M \subseteq a^{n-1} M \subseteq \cdots \subseteq a^2 M \subseteq aM \subseteq M$$

is isomorphic to M / aM, and moding out this chain by $a^n M$ gives the chain

$$0 \subseteq a^{n-1}M / a^n M \subseteq \cdots \subseteq aM / a^n M \subseteq M / a^n M$$

and (1) follows. Similarily, $\ell_A(E / a^n E) = n \cdot \ell_A(E / aE)$. But $E \cong A^r$, so $E / aE \cong (A / aA)^r$ and thus

$$\ell_A(E / aE) = r \cdot \ell_A(A / aA)$$

Hence (2) and the claim follows.

From $(*)$ we have $\ell_A(C / a^n C) \leqslant \ell_A(C)$, so the claim gives

$$\ell_A(M / aM) \leqslant r \cdot \ell_A(A / aA) + \frac{1}{n} \cdot \ell_A(C / a^n C)$$

and Case 1 follows by taking n large enough.

<u>Case 2</u> General M. Consider the family $\{T_\alpha\}_{\alpha\in\Lambda}$, where $T_\alpha := M_\alpha/(M_\alpha \cap aM)$ and $M_\alpha \subseteq M$ is a finitely generated submodule, so

$$M_\alpha/aM_\alpha \longrightarrow T_\alpha \longrightarrow 0$$

is a right exact sequence and thus

$$\ell_A(T_\alpha) \leqslant \ell_A(M_\alpha/aM_\alpha) \overset{C1}{\leqslant} r\cdot \ell_A(A/aA),$$

for each $\alpha\in\Lambda$. But $\{T_\alpha\}_{\alpha\in\Lambda}$ is a directed family of A-modules s.t.

$$M/aM = \bigcup_{\alpha\in\Lambda} T_\alpha .$$

By Lemma 1.1 $\ell_A(M/aM) \leqslant r\cdot \ell_A(A/aA)$ and the lemma follows $\square$