

Lecture 2 The Krull - Akizuki theorem

Theorem 1.1 (Krull - Akizuki) Let A be a 1-dim'l Noetherian domain with field of fractions K , let L/K be a field ext'n s.t. $[L:K] < \infty$ and B a ring s.t.



Then

- (1) $\ell_A(B/J) < \infty$, for each nonzero ideal $J \subseteq B$
- (2) B is Noetherian & $\dim(B) \leq 1$

Proof

(1): WLOG we may assume that L is the field of fractions of B so that B is a torsion free A -module of rank $r := [L:K]$. Consider a nonzero ideal $J \subseteq B$.

Claim There is a nonzero element $a \in J \cap A$.

Proof of claim

There is nonzero $b \in J$ and we may pick $a_0, \dots, a_m \in A$ s.t.

$$a_m b^m + \dots + a_1 b + a_0 = 0$$

with m of minimal degree. But $B \subseteq L$, so B is a domain and thus $a_0 \in J \cap A \neq 0$. The

claim follows.

Therefore

$$l_A(B/J) \leq l_A(B/aB) \leq \overbrace{r \cdot l_A(A/aA)}^{L1.4} < \infty,$$

where the first inequality comes from the right exact sequence $B/aB \rightarrow B/J \rightarrow 0$ and (1) follows.

(2): We have the left exact sequence $0 \rightarrow J/aB \rightarrow B/aB$, hence

$$l_B(J/aB) \leq l_A(J/aB) \leq \overbrace{l_A(B/aB)}^{L1.4} < \infty.$$

Therefore $\ell_B(\mathcal{I}/aB) < \infty$, so $\mathcal{I}/aB$ is a Noetherian B -module.

But we have a short exact sequence of B -modules

$$0 \rightarrow aB \rightarrow \mathcal{I} \rightarrow \mathcal{I}/aB \rightarrow 0$$

Also, as A is a domain and $a \neq 0$ we have a B -module isomorphism

$$B \xrightarrow{\sim} aB$$

$$b \mapsto ab$$

So aB is a Noetherian B -module as well. Thus $\mathcal{I}$ is Noetherian B -module.

Finally, if $\dim(B) > 1$ then $\exists \mathcal{P}_1, \mathcal{P}_2 \in \text{Spec}(B)$ s.t.

$$0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq B.$$

Thus the domain $B/\mathfrak{p}_1$ is not a field, so by Lemma 1.2

$$l_B(B/\mathfrak{p}_1) = \infty,$$

which contradicts (1). Hence $\dim(B) \leq 1$ and (2) follows $\square$

We say that A is a *Dedekind* ring if A is Noetherian, integrally closed, and s.t. $\dim(A) = 1$.

Corollary 1 If in $(\star)$ we further assume that A is integrally closed (so that A is Dedekind) and that B is the integral closure of A in L , then B is Dedekind.

Remark Suppose that L/K is purely inseparable. So a suitable power of Frobenius, $\varphi(x) = x^{p^r}$ (where $p = \text{char}(K) > 0$ and suitable $r \in \mathbb{Z}_{\geq 1}$) gives

$$\varphi \begin{array}{c} \mathcal{O}_{R,L} \\ \uparrow \iota \\ R \end{array} \longmapsto \varphi^* \begin{array}{c} \text{Spec}(\mathcal{O}_{R,L}) \\ \downarrow \iota^* \\ \text{Spec}(R) \end{array}$$

As $\mathcal{O}_{R,L}/R$ is integral, ι^* is a **surjective morphism**. Moreover, ι^* is in fact an **isomorphism**, i.e.

$$\varphi^* \circ \iota^* = 1_{\text{Spec}(\mathcal{O}_{R,L})}$$

Indeed, for all $\mathfrak{P} \in \text{Spec}(\mathcal{O}_{R,L})$ we have the equality

$$\varphi^{-1}(\mathfrak{P} \cap R) = \mathfrak{P}.$$

$(\subseteq)$: If $x \in \varphi^{-1}(\mathfrak{P} \cap R)$ then $x^{p^r} \in \mathfrak{P} \cap R$. So $x \in \mathfrak{P}$ as $\mathfrak{P}$ is prime.

$(\supseteq)$: If $x \in \mathfrak{P}$ then $x^{p^r} \in \mathfrak{P}$, but $x^{p^r} \in R$, so $x \in \varphi^{-1}(\mathfrak{P} \cap R)$.

Prop'n 1.3 Let R be a Noetherian domain of dimension one. For each nonzero ideal $a \subseteq R$ we have a unique product decomposition into primary ideals

$$a = \mathfrak{q}_1 \cdots \mathfrak{q}_g$$

where $\text{rad}(\mathfrak{q}_1), \dots, \text{rad}(\mathfrak{q}_g) \in \text{Spec}(R)$ are distinct.

Proof

As we assumed that R is Noetherian, we have the minimal primary decomposition

$$a = \bigcap_i \mathfrak{q}_i, \quad *$$

where $p_i := \text{rad}(\mathfrak{q}_i) \in \text{Spec}(R)$, for each $i \in \{1, \dots, g\}$. But R is a domain

with Krull dimension $\dim(R) = 1$. Thus $p_1, \dots, p_g \in \text{Spec}(R)$ are distinct maximal ideals and thus for each $i, j \in \{1, \dots, g\}$ s.t. $i \neq j$ we have

$$p_i + p_j = R,$$

so

$$\mathfrak{q}_i + \mathfrak{q}_j = R.$$

Claim More generally, let $I_1, \dots, I_n \subseteq R$ be ideals s.t. $j \neq i$ implies that $I_i + I_j = R$, for all $i, j \in \{1, \dots, n\}$. Then

$$\bigcap_{i=1}^n I_i = \prod_{i=1}^n I_i$$

Proof of claim — by induction on n

Case $n = 2$: So we have

$$x_1 + x_2 = 1,$$

where $x_1 \in I_1$ and $x_2 \in I_2$. Then $\forall y \in I_1 \cap I_2$

$$yx_1 + yx_2 = y.$$

But $yx_1 \in I_1 I_2$ and $yx_2 \in I_1 I_2$, hence

$$I_1 \cap I_2 \subseteq I_1 I_2.$$

The reverse inclusion is clear, so the equality follows.

Case $n > 2$: $I'_{n-1} := I_{n-1} \cap I_n$. So by Case 1, $I'_{n-1} := I_{n-1} I_n$.

Now we apply the inductive hypothesis to the $n-1$ ideals $I_1, \dots, I_{n-2}, I'_{n-1}$,

$$\bigcap_{i=1}^n I_i = I_1 \cap \dots \cap I_{n-2} \cap I'_{n-1} = I_1 \dots I_{n-2} I'_{n-1} = \prod_{i=1}^n I_i$$

and the claim follows.

The propⁿ follows if we apply the claim to $(*)$ $\square$