Lecture 11 The Bernoulli numbers $B_{k,\psi}$

The *Bernoulli numbers* $B_0, B_1, B_2, \ldots$ are defined by

$$f(X) := \frac{T}{e^T - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} T^n$$

and the *Bernoulli polynomials* by

$$F(T,X) := \frac{T}{e^T - 1} e^{TX} = \sum_{n=0}^{\infty} \frac{B_n(X)}{n!} T^n$$

we have

$$F(T,X) = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} T^n \right) \left(\sum_{m=0}^{\infty} \frac{X^m}{m!} T^m \right)$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} B_i X^{n-i} \frac{T^n}{n!} .$$

Therefore

$$B_n(X) = \sum_{i=0}^{n} \binom{n}{i} B_i X^{n-i}$$

We have the identity

$$F(T,X) - F(T,X-1) = Te^{T(X-1)}$$

and thus

$$B_{n+1}(X) - B_{n+1}(X-1) = (n+1)X^n$$

and

$$\frac{B_{n+1}(X) - B_{n+1}(X-1)}{n+1} = X^n.$$

Summing this for $X = 1, 2, 3, \dots, k$ we get for each $n \in \mathbb{Z}_{\geqslant 1}$ we get

$$S_n(k) := 1^n + 2^n + 3^n + \cdots + k^n = \frac{B_{n+1}(k) - B_{n+1}(0)}{n+1}$$

This is known as *Faulhaber's formula*. For example, for $n = 4$ we have

$$B_0 = 1, \quad B_1 = \frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30};$$

$$S_4(k) = \frac{1}{5}\left(B_0 k^5 + 5 B_1 k^4 + 10 B_2 k^3 + 10 B_3 k^2 + 5 B_4 k\right) =$$

$$= \frac{1}{5} k^5 + \frac{1}{2} k^4 + \frac{1}{3} k^3 - \frac{1}{30} k.$$

Given $N \in \mathbb{Z}_{\geq 1}$, a *Dirichlet character* χ *of modulus* N is a periodic function $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$ with period N s.t.

(i) $\forall\, n, m \in \mathbb{Z} : \quad \chi(nm) = \chi(n)\,\chi(m)$

(ii) $\forall\, n \in \mathbb{Z} : \quad \chi(n) \neq 0 \iff (n, N) = 1.$

In particular, a Dirichlet character induces a group homomorphism

$$(\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

The nth Bernoulli number attached to a character ψ of conductor N is[1]

$$B_{k,\psi} = N^{k-1} \sum_{i=0}^{N-1} \psi(i) B_k\left(\frac{i}{N}\right)$$

Note that $B_1(X) = X - \frac{1}{2}$. So ψ odd implies that

$$B_{1,\psi} = \frac{1}{N} \sum_{i=0}^{N-1} \psi(i)\, i$$

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1 Cf. p. 250 in Lang S., *Introduction to Modular Forms*, Grundlehren der mathematischen Wissenschaften 222, Second Corrected Printing 1995.

Hence for the nontrivial character ψ of conductor $N = 4$ we have

$$B_{1,\psi} = \frac{1}{4}\left(\psi(1) + \psi(2)\,2 + \psi(3)\,3 \right) = -\frac{1}{2}.$$

For primes $p \equiv 1 \pmod 4$ we have $\chi(a) = \left(\frac{a}{p}\right)$ is a nontrivial even Dirichlet character.

p	$\frac{1}{4} B_{2,\chi}$
5	$\frac{1}{5}$
13	1
17	2