Lecture 12 Some modular units and Eisenstein series

The *Dedekind η-function* is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3}{2}n^2 - \frac{1}{2}n + \frac{1}{24}},$$

where $q = e^{2\pi i\tau}$ and $\tau \in \mathcal{H} := \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$

We have for each $N \in \mathbb{Z}_{>1}$ s.t. $N-1 \mid 24$ i.e.

$$N \in \{2, 3, 4, 5, 7, 9, 13\}$$

the modular unit

$$f_N := \frac{\eta(\tau)}{\eta(N\tau)} = q^{\frac{1}{24}(1-N)} \prod_{n=1}^{\infty} (1-q^n)^{\xi(n)}$$

Hence

$$\frac{f_N'}{f_N} = 2\pi i \left(\frac{1}{24}(1-N) - \sum_{n=1}^{\infty} \sigma_{1,\xi}(n) q^n \right)$$

$$c_N \frac{f_N'}{f_N} = 1 + \frac{24}{N-1} \sum_{n=1}^{\infty} \sigma_{1,\xi}(n) q^n$$

Moreover, we have the Eisenstein series of weight $k \in 2\mathbb{Z}_{\geq 1}$

$$E_{k,\chi} = 1 - \frac{2k}{B_{\chi,k}} \sum_{n=1} \sigma_{k-1,\chi}(n) q^n,$$

where χ is an even Dirichlet character of conductor N. For $k = 2$ it is the logarithmic derivative of

$$u_\chi(\tau) = q^{\frac{1}{4} B_{2,\chi}} \prod_{n=1}^{\infty} (1 - q^n)^{\chi(n)}$$

Consider an odd prime $p \equiv 1 \pmod 4$ so that the Legendre character

$$\chi(\cdot) = \left(\frac{\cdot}{p}\right)$$

is even. For $p = 5$ we have

$$u_\chi = \cfrac{q^{\frac{1}{5}}}{1 + \cfrac{q}{1 + \cfrac{q^2}{\ddots}}} \longrightarrow \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}} = \frac{1}{\varphi}$$

as $q \to 1$.

Fix $m \in \mathbb{Z}_{\geq 1}$. For each $n \in \mathbb{Z}_{\geq 0}$ let

$$r(n) := \# \{ (x_1, \dots, x_m) \in \mathbb{Z}^m \mid n = x_1^2 + \cdots + x_m^2 \}$$

For $m = 1$ we have

$$r_1(n) = \begin{cases} 2, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise.} \end{cases}$$

For $m = 2$ we have the following table.

n	$(x_1, x_2) \in \mathbb{Z}^2$ s.t. $n = x_1^2 + x_2^2$	$r_2(n)$
0	$(0,0)$	1
1	$(0,1), (0,-1), (1,0), (-1,0)$	4
2	$(1,1), (1,-1), (1,-1), (-1,-1)$	4
3		0
4	$(0,2), (0,-2), (2,0), (-2,0)$	4
5	$(1,2), (1,-2), (2,1), (-2,1), (-1,2), (-1,-2), (2,-1), (-2,-1)$	8

For $m = 3$ we have the following table.

n	$(x_1, x_2, x_3) \in \mathbb{Z}^2$ s.t. $n = x_1^2 + x_2^2 + x_3^2$	$r_3(n)$
0	$(0,0,0)$	1
1	$(1,0,0), (0,1,0), (0,0,1), (-1,0,0), (0,-1,0), (0,0,-1)$	6
2	$(1,1,0), \ldots$	12
3	$(1,1,1), \ldots$	8
4	$(2,0,0), \ldots$	6
5	$(2,1,0), \ldots$	24

For $m = 4$ we have the following table.

n	$(x_1, x_2, x_3, x_4) \in \mathbb{Z}^2$ s.t. $n = x_1^2 + x_2^2 + x_3^2 + x_4^2$	$r_4(n)$
0	$(0,0,0,0)$	1
1	$(1,0,0,0), \ldots$	8
2	$(1,1,0,0), \ldots$	24
3	$(1,1,1,0), \ldots$	32
4	$(2,0,0,0), \ldots (1,1,1,1), \ldots$	24
5	$(2,1,0,0), \ldots$	48

We have the generating function for $r_m(n)$

$$\vartheta^m = \sum_{n=0}^{\infty} r_m(n)\, q^n \qquad \star$$

where

$$\vartheta = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

It turns out that for $m \in \mathbb{Z}_{\geqslant 1}$ the series $(\star)$ is a modular form of weight $k = \frac{m}{2}$. This theory shows that if we define

$$\sigma_{\alpha,s}(n) = \sum_{d\mid n} \alpha(d)\, d^s$$

then

	weight	level	
$\theta^2 = 1 + 4 \sum_{n=1}^{\infty} \sigma_{\psi,0}(n)\, q^n$	1	4	new
$\theta^4 = 1 + 8 \sum_{n=1}^{\infty} \sigma_{\xi,1}(n)\, q^n$	2	4	old

where

n	$\cdots$	-3	-2	-1	0	1	2	3	4	5	6	$\cdots$
$\psi(n)$	$\cdots$	1	0	-1	0	1	0	-1	0	1	0	$\cdots$

n	$\cdots$	-3	-2	-1	0	1	2	3	4	5	6	$\cdots$
$\xi(n)$	$\cdots$	1	1	1	0	1	1	1	0	1	1	$\cdots$

We have the Einsenstein series of weight $k=1$ and level N,

$$E_{1,\psi} = 1 - \frac{2}{B_{1,\psi}} \sum_{n=1}^{\infty} \sigma_{1,\psi}(n)\, q^n$$

where $B_{1,\psi}$ is the 1^{st} Bernoulli number attached to an odd Dirichlet character.

For

$$\psi(n) = \left(\frac{-4}{n}\right)$$

we have

$$\vartheta^2 = E_{1,\psi}$$

Here

$$\sigma_{s,\chi}(n) := \sum_{d|n} \chi(d)\, d^{s}$$

where χ is a suitable Dirichlet character and $B_{k,\chi}$ is the k-th Bernoulli number attached to χ.