Lecture 2 Syntactic properties of truth

Following Manin[1], consider the following tautologies

A0. $P \Rightarrow P$

A1. $P \Rightarrow (Q \Rightarrow P)$

A2. $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))$

A3. $(\neg Q \Rightarrow \neg P) \Rightarrow ((\neg Q \Rightarrow P) \Rightarrow Q)$

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1 Manin, Yu. I., *A Course in Mathematical Logic for Mathematicians*, GTM, 53, Springer-Verlag.

B1. $\neg(\neg P) \Rightarrow P$, $P \Rightarrow (\neg(\neg P))$

B2. $\neg P \Rightarrow (P \Rightarrow Q)$

and the "quantifier axioms"

(a) $\forall x (P \Rightarrow Q) \Rightarrow (P \Rightarrow (\forall x (Q)))$

(b) $(\forall x (\neg P)) \Leftrightarrow (\exists x (P))$

(c) $(\forall x (P)) \Rightarrow P(y)$, where y is free for x in P.

Here P, Q, and R denote formulas. We have the rule of deduction *Modus Ponens*

$$\frac{\begin{matrix} P \\ P \Rightarrow Q \end{matrix}}{Q}$$

Given a predicate language L and universe of discourse Ω, let $T_\Omega(L)$ denote the set of true formulas. We may ask whether there is a set of closed formulas $\mathcal{E}$ such that every formula in $T_\Omega(L)$ may be obtained by recursively applying the above starting from $\mathcal{E}$ e.g. the axioms of Euclidean geometry with universe of discourse $\Omega = \mathbb{R}^2$.

From the work of Hilbert, Gödel, and others it became clear the importance of the formalisation of the language of mathematics. Their efforts lead to the study of mathematical logic. In particular, in 1931 Gödel proved his profund incompleteness theorem[1]. It's an open question whether the Riemann Hypothesis is true, but not a logical consequence of Set Theory (i.e. Zermelo - Frenkel + Choice Axiom).

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1 Gödel, K., *Über formal unentscheidbare Sätze der Principia Mathematica und Verwandter systeme I*, Monatshefte für Mathematik, 1931.

Quite remarkably, the RH is equivalent to the following statement expressed in terms of basic arithmetic

$$\forall \varepsilon > 0 \; \exists x \; \forall y \left(y > x \Rightarrow \left(\left| \sum_{n=1}^{y} \mu(n) \right| < y^{\frac{1}{2}+\varepsilon} \right) \right)$$

Here μ is the *Möbius function*

$$\mu(n) := \begin{cases} (-1)^{\ell}, & \text{if } n = p_1 \cdots p_\ell, \quad p_1 < \cdots < p_\ell \text{ are primes,} \\ 0, & \text{otherwise.} \end{cases}$$